ランダム行列の２次漸近挙動のデータ解析への応用

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A part of this talk is base on the joint works with
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Random Matrices, Free Probability, and Machine Learning
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Random matrices

In this talk, we deal with two kinds of random matrices

- the Wishart matrices (the sample covariance matrix),
- the Wigner matrix (in general, not GUE or GOE),

being *real symmetric* or *hermitian* matrices, depending on the entries in $\mathbb{R}$ or $\mathbb{C}$.

Plan of talk

1. The first and second order limits for random matrices
   - Case of the Wishart matrices
   - Case of the compound Wishart matrices
   - Examples of explicit formulas of fluctuations
2. An application to data analysis
   - Random matrices with dependent entries
   - Application to time series
3. Case of Wigner matrices ($\leftarrow$ if we have time)
   - Diagonally deformed GUE (not unitary invariant)
   - Fluctuation formula

The first and second order limits

Empirical distribution of random matrix

The empirical distribution

Suppose that $\mathbf{X}_N$ is an $N \times N$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N \in \mathbb{C}$, the empirical distribution of the eigenvalues or the empirical spectral distribution (ESD) of $\mathbf{X}_N$, is

$$d\mu_{\mathbf{X}_N}(x) = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i}(x)$$

where δ_y denotes the Dirac mass at y.

Empirical distribution function

The empirical distribution function

If $\mathbf{X}_N$ is Hermitian, so that the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$ are real, we can define the empirical distribution function of $\mathbf{X}_N$ as

$$F_{\mathbf{X}_N}(x) = \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}_{\lambda_i \leq x} \quad \text{for} \quad x \in \mathbb{R}.$$

In RMT, the ESD of a random matrix plays a central role in studying the properties of the spectrum, that is, the asymptotic behavior of the ESD (random distribution).

The limit distribution

The question one can ask about the ESD is whether, after appropriate normalization of the matrix, this random distribution converges to a certain probability distribution in an appropriate sense as the dimension of the random matrix goes to infinite.

If the above probability distribution exists, it is called
the limit distribution of the random matrices.

The famous limit laws

The Marcenko-Pastur law (the free Poisson law)

$$\pi_\lambda(dx) = \frac{\sqrt{-(x-\lambda_-)(x-\lambda_+)}}{2\pi\,\lambda\,x}\,\mathbf{1}_{[\lambda_-,\lambda_+]}(x)\,dx + \max\{0, 1-\lambda\}\delta_0(x),$$

where $\lambda_\pm = \left(1 \pm \sqrt{\lambda}\right)^2$, gives an answer for Wishart matrices.

The semicircle law

$$\mu(dx) = \frac{1}{2\pi}\sqrt{4-x^2}\,\mathbf{1}_{[-2,2]}(x)\,dx$$

is an answer in the context of Wigner matrices.

The second order limit distribution

The first order limit

Consider a sequence of random matrices $\mathcal{A} = \left(\mathbf{X}_N\right)_{N\in\mathbb{N}}$ and look the random variables $\left\{\mathrm{tr}\left(\mathbf{X}_N^k\right)\right\}$, where tr is the normalized trace.

Limit distribution and moments

The limit distribution means that the limit

$$\alpha_k = \lim_{N\to\infty} \mathrm{E}\left[\mathrm{tr}\left(\mathbf{X}_N^k\right)\right],$$

exist for all $k \in \mathbb{N}$. We call α_k is *the kth moment*.

Remark.
We can find that in many cases the fluctuation $\mathrm{tr}\big(\mathbf{X}_N^k\big) - \alpha_k$ is asymptotically Gaussian of order $\frac{1}{N}$, that is, the random variable

$$N\big(\mathrm{tr}\big(\mathbf{X}_N^k\big) - \alpha_k\big) = \mathrm{Tr}\big(\mathbf{X}_N^k - \alpha_k \mathbf{1}\big)$$

converges to a centered Gaussian random variable as $N \to \infty$, where Tr denotes the unnormalized trace (simple sum of the diagonal entries).

The main information about the fluctuations of Gaussian random variables $\big\{\mathrm{Tr}\big(\mathbf{X}_N^k - \alpha_k \mathbf{1}\big)\big\}_{k \geq 1}$ is given by the covariances.

Fluctuations

$$\begin{aligned}\alpha_{p,q} &= \lim_{N\to\infty} \mathrm{Cov}\big(\mathrm{Tr}(\mathbf{X}_N^p), \mathrm{Tr}(\mathbf{X}_N^q)\big) \\ &= \lim_{N\to\infty} \mathrm{E}\big[\mathrm{Tr}\big(\mathbf{X}_N^p - \alpha_p \mathbf{1}\big) \cdot \mathrm{Tr}\big(\mathbf{X}_N^q - \alpha_q \mathbf{1}\big)\big].\end{aligned}$$

We call these quantities *the fluctuation moments* (or simply, *fluctuation*).

One of purposes of this talk is to explain some explicit formulas for the fluctuation of some random matrices.

The Gaussianity

The classical Gaussianity can be characterized as that all the higher classical cumulants other than the first and second will vanish.

Since the expectation and the covariance are corresponding to the first and second classical cumulants, respectively, the following definition will arise naturally for the case of Gaussian fluctuations.

Definition (The second order limit distribution)

Let $(X_N)_{N\in\mathbb{N}}$ be a sequence of $N \times N$ random matrices. We say that it has a *second order limit distribution* if for $p, q \geq 1$ all the limits

$$\alpha_p = \lim_{N\to\infty} \kappa_1\big(\mathrm{tr}(X_N^p)\big)$$

and

$$\alpha_{p,q} = \lim_{N\to\infty} \kappa_2\big(\mathrm{Tr}(X_N^p), \mathrm{Tr}(X_N^q)\big)$$

exist, and if for all $r \geq 3$ and all $p_1, p_2, \ldots, p_r \in \mathbb{N}$

$$\lim_{N\to\infty} \kappa_r\big(\mathrm{Tr}(X_N^{p_1}), \mathrm{Tr}(X_N^{p_2}), \ldots, \mathrm{Tr}(X_N^{p_r})\big) = 0.$$

Case of the Wishart matrices

The limit distribution of *complex* Wishart matrix

Wishart matrix

Let $\mathbf{G}_{N,M}$ be an $N \times M$ matrix, the entries X_{ij} of which are independent *complex* centered ($\mathrm{E}(X_{i,j}) = 0$) Gaussian random variables with variance 1 ($\mathrm{Var}(X_{i,j}) = 1$), and let

$$\mathbf{W}_N = \frac{1}{N}\mathbf{G}_{N,M}\mathbf{G}_{N,M}^*.$$

The $N \times N$ random matrix $\mathbf{W}_N$ is called *complex* <u>*Wishart matrix*</u>.

Consider the Marchenko-Pastur limit, that is, we take the limit $N \to \infty$, and here M also goes to infinite with the asymptotic ratio $M/N \to \lambda > 0$, then the empirical spectral distribution $\mu_{\mathbf{W}_N}$ converges in a.s. to π_λ,

$$d\pi_\lambda(x) = \frac{\sqrt{-(x-\lambda_-)(x-\lambda_+)}}{2\pi\,\lambda\,x}\,\mathbf{1}_{[\lambda_-,\lambda_+]}(x)\,dx + \max\{0, 1-\lambda\}\delta_0(x).$$

The moments of the Marchenko-Pastur law

For the limit distribution (the Marchenko-Pastur law), the moment can be expressed as

$$\alpha_n = \sum_{\pi \in NC(n)} \lambda^{b(\pi)}$$

where $NC(p)$ is the set of the non-crossing partitions of $[n]$ and $b(\pi)$ denotes the number of blocks in the partition π.

The non-crossing partitions and the partition statistic $b(\pi)$ are explained later.

The explicit formula of the moment is known as $\alpha_n = \sum_{k=1}^{n} N(n,k)\,\lambda^k$,

where $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{n-k}$ is called the Narayana number.

Definition (Partirions)

For an ordered set $S = \{1, 2, \ldots, n\}$, $\pi = \{B_1, B_2, \ldots, B_k\}$ is said to be a *partition of* S if

$$B_i \neq \phi \ \text{ and } \ B_i \cap B_j = \phi \ \text{ with } \ \bigcup_{i=1}^{k} B_i = S.$$

An element $B \in \pi$ is called *block* and $|B|$ denotes the number of elements in B, and $|\pi|$ stands the number of blocks in the partition π.

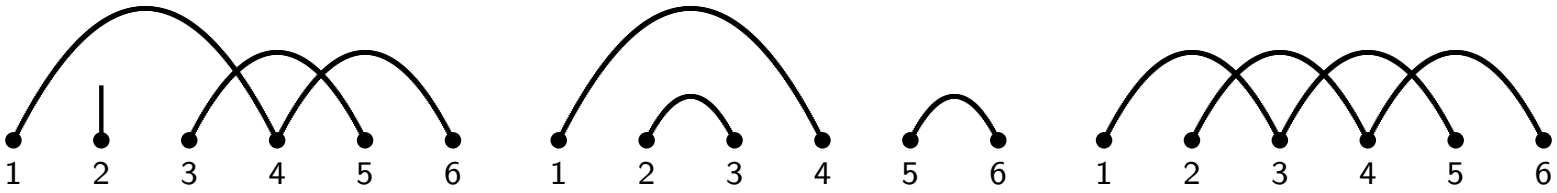
We denote the set of all partitions of $\{1, 2, \ldots, n\}$ by $\mathcal{P}(n)$.

Examples of set partitions

n	type	partitions
1	$[1]$	$\{\{1\}\}$
2	$[2]$	$\{\{1,2\}\}$
	$[1^2]$	$\{\{1\},\{2\}\}$
3	$[3]$	$\{\{1,2,3\}\}$
	$[2,1]$	$\{\{1,2\},\{3\}\},\ \{\{1,3\},\{2\}\},\ \{\{2,3\},\{1\}\}$
	$[1^3]$	$\{\{1\},\{2\},\{3\}\}$
4	$[4]$	$\{\{1,2,3,4\}\}$
	$[3,1]$	$\{\{1,2,3\},\{4\}\},\ \{\{1,2,4\},\{3\}\},\ \{\{1,3,4\},\{2\}\},\ \{\{2,3,4\},\{1\}\}$
	$[2^2]$	$\{\{1,2\},\{3,4\}\},\ \{\{1,3\},\{2,4\}\},\ \{\{1,4\},\{2,3\}\}$
	$[2,1^2]$	$\{\{1,2\},\{3\},\{4\}\},\ \{\{1,3\},\{2\},\{4\}\},\ \{\{1,4\},\{2\},\{3\}\},$
		$\{\{2,3\},\{1\},\{4\}\},\ \{\{2,4\},\{1\},\{3\}\},\ \{\{3,4\},\{1\},\{2\}\}$
	$[1^4]$	$\{\{1\},\{2\},\{3\},\{4\}\}$

Graphical representation of partitions

We arrange $\{1, 2, \ldots, n\}$ on a line and connect elements in the same block by arcs successively, in the upper half plain. Here a block of size 1 (singleton) will not be connected to any other point.



$\{\{1, 4, 6\}, \{2\}, \{3, 5\}\}$ $\{\{1, 4\}, \{2, 3\}, \{5, 6\}\}$ $\{\{1, 3, 5\}, \{2, 4, 6\}\}$

Definition (Non-crossing partitions)

For a partition $\pi = \{B_1, B_2, \ldots, B_k\} \in \mathcal{P}(n)$, if there exist two blocks B_i, B_j $(i \neq j)$ and elements $b_1, b_2 \in B_i$, $c_1, c_2 \in B_j$ such that $b_1 < c_1 < b_2 < c_2$, then π is called *crossing*.

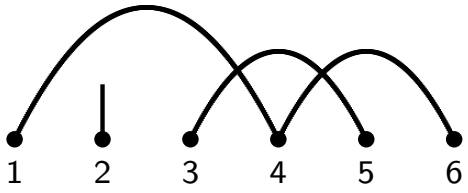
If a partition π does not have any such a crossing, then π is called *non-crossing*.

We denote the set of all non-crossing partitions of $\{1, 2, \ldots, n\}$ by $\mathcal{NC}(n)$.

Remark.

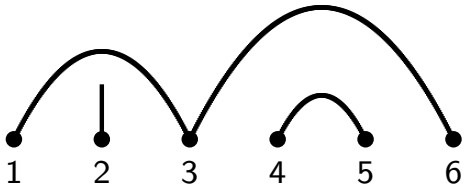
$\mathcal{NC}(n) \subset \mathcal{P}(n)$: sub lattice

Graphical meaning of non-crossing



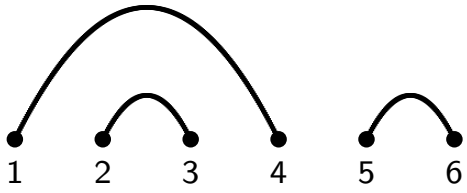
$\{\{1,3,5\},\{2\},\{4,6\}\}$

crossing



$\{\{1,3,6\},\{2\},\{4,6\}\}$

non-crossing



$\{\{1,4\},\{2,3\},\{5,6\}\}$

non-crossing

Examples of non-crossing partitions

n	type	partitions
1	$[1]$	$\{\{1\}\}$
2	$[2]$	$\{\{1,2\}\}$
	$[1^2]$	$\{\{1\},\{2\}\}$
3	$[3]$	$\{\{1,2,3\}\}$
	$[2,1]$	$\{\{1,2\},\{3\}\}$, $\{\{1,3\},\{2\}\}$, $\{\{2,3\},\{1\}\}$
	$[1^3]$	$\{\{1\},\{2\},\{3\}\}$
4	$[4]$	$\{\{1,2,3,4\}\}$
	$[3,1]$	$\{\{1,2,3\},\{4\}\}$, $\{\{1,2,4\},\{3\}\}$, $\{\{1,3,4\},\{2\}\}$, $\{\{2,3,4\},\{1\}\}$
	$[2^2]$	$\{\{1,2\},\{3,4\}\}$, $\cancel{\{\{1,3\},\{2,4\}\}}$, $\{\{1,4\},\{2,3\}\}$
	$[2,1^2]$	$\{\{1,2\},\{3\},\{4\}\}$, $\{\{1,3\},\{2\},\{4\}\}$, $\{\{1,4\},\{2\},\{3\}\}$, $\{\{2,3\},\{1\},\{4\}\}$, $\{\{2,4\},\{1\},\{3\}\}$, $\{\{3,4\},\{1\},\{2\}\}$
	$[1^4]$	$\{\{1\},\{2\},\{3\},\{4\}\}$

non-crossing partitions appear for $n \geq 4$

The second order asymptotics of *complex* Wishart matrix

The fluctuations of Wishart matrix [Mingo,Speicher (2006)]

$$\alpha_{p,q} = \lim_{N\to\infty} \mathrm{Cov}\big(\mathrm{Tr}(\mathbf{X}_N^p), \mathrm{Tr}(\mathbf{X}_N^q)\big) = \sum_{\pi\in S_{annNC}(p,q)} \lambda^{c(\pi)},$$

where $S_{annNC}(p,q)$ is the set of the non-crossing (p,q)-annular permutations and $c(\pi)$ denotes the number of cycles in the permutation π.

Non-crossing (p,q)-annular permutations was introduced by [Mingo, Nica (2004)], which will be explained below:

Annulus

Annulus and Cycle

Let p and q be positive integers. A (p, q)-*annulus* is an annulus in which $1, 2, \ldots, p$ are arranged in clockwise order on the external circle, and $p+1, p+2, \ldots, p+q$ are arranged in anti-clockwise order on the internal circle.

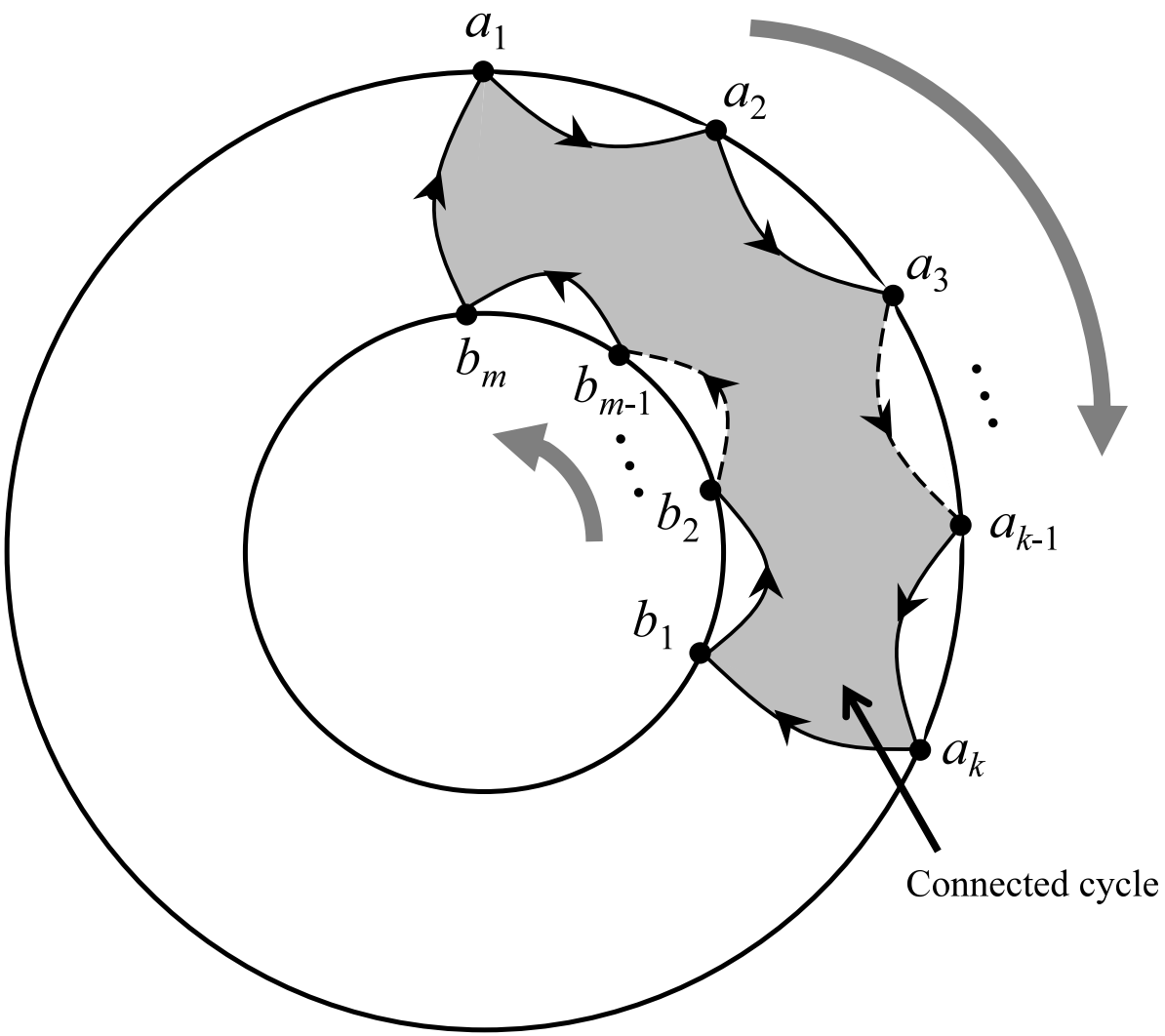
Let $(c_1, c_2, \ldots, c_k)$ be a *cycle of order* k with $c_i \in [p+q]$. We will represent this cycle inside (p, q)-annulus by drawing an arrow from c_i to c_{i+1} $(i = 1, 2, \ldots, k)$ where $c_{k+1} = c_1$ with an orientation in clockwise.

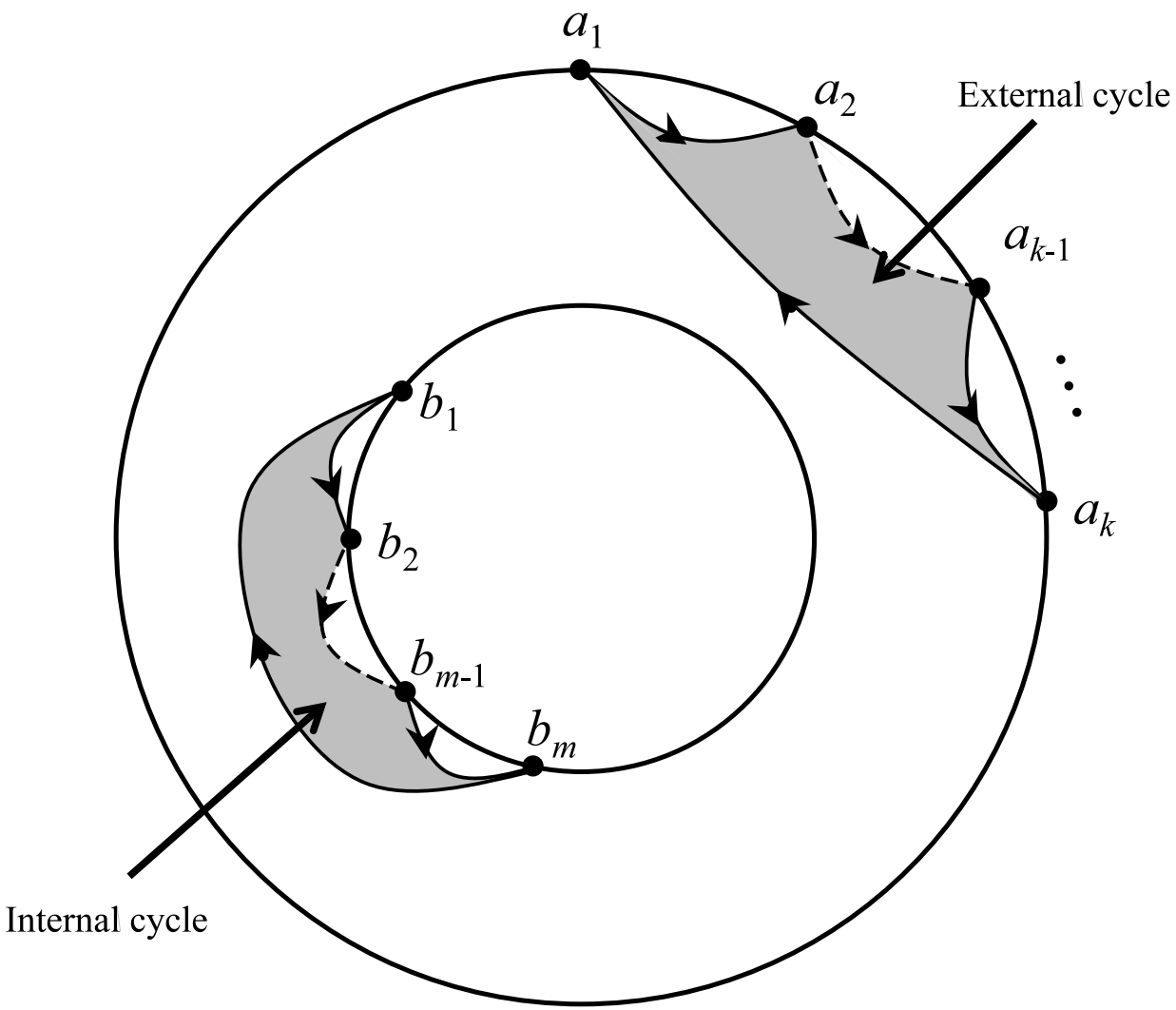
Cycles

Cycles

A cycle is called *internal* (resp.*external*) if all of whose elements are in the internal (resp. external) circle.

A cycle is called *conected* if it is a cycle which contains both elements in the inner and the outer circles.


a_1
a_2
a_3
$a_{k\text{-}1}$
a_k
b_m
$b_{m\text{-}1}$
b_2
b_1
Connected cycle

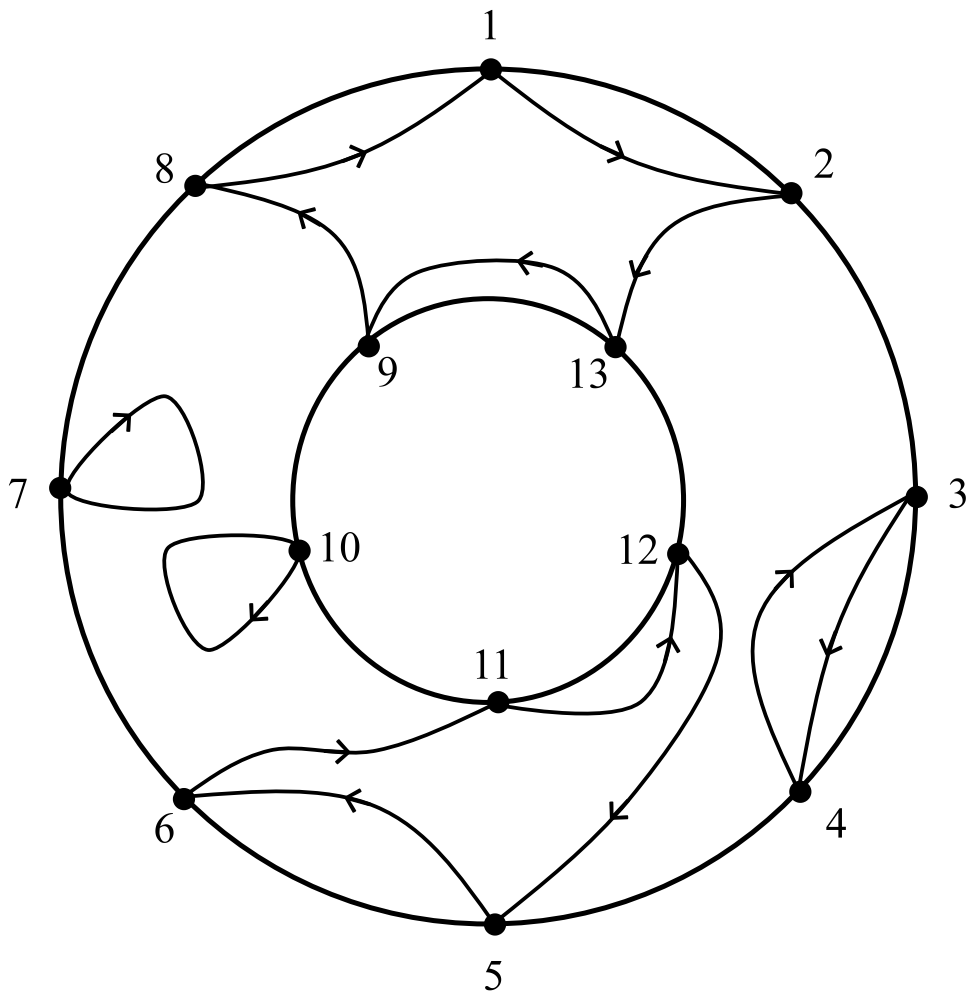

a_1
a_2
External cycle
$a_{k\text{-}1}$
a_k
b_1
b_2
$b_{m\text{-}1}$
b_m
Internal cycle

Definition (Non-crossing annular permutations)

Non-crossing (p, q)*-annular permitation* is a permutation of $[p + q]$ for which we can draw its oriented cycles inside (p, q)-annulus in non-crossing way and it has *at least one* connected cycle. We denote by $S_{annNC}(p, q)$ the set of all non-crossing (p, q)-annular permitations.

If there is no connected cycle inside (p, q)-annulus on drawing the cycles in the permutation π on $[1, p + q]$, then it is essentially a disjoint union of two non-crossing permutations on $[1, p]$ and $[p + 1, p + q]$. Thus we will consider only connected annular non-crossing permutations for the set $S_{annNC}(p, q)$.

Example of $S_{annNC}(8,5)$



$$\pi = (1,2,13,9,8)(3,4)(5,6,11,12)(7)(10)$$

The number of cycles

Let π be the permutation on $[1, p+q]$. Then $[1, p+q]$ is partitioned into the orbits of π. The orbit of π is called *cycle*, and we will usually write a permutation in the cycle notation. Although it is customary to omit the orbits with one element from the cycle notation, we will not remove any of them.

The number of cycles

$$c(\pi) = \text{the number of cycles (orbits) in } \pi,$$

which is one of the fundamental permutation statistics.

For a cycle (an orbit) C in π, we denote by $|C|$ the order of the cycle (the length of the orbit) C.

Remarks

- It is obvious that the map changing each cycle to a block does not always give a one-to-one correspondence between annular non-crossing permutations and annular non-crossing partitions.

  For instance, two $(2,1)$-annular non-crossing permutations $(1,2,3)$ and $(2,1,3)$ are sent to the partition with only one block $\{1,2,3\}$.

- It is known, however, that if there are at least two connected cycles, then this map becomes a bijection.

In case of $\lambda = 1$ for complex Wishart matrix

$$\alpha_{p,q} = \left|S_{annNC}(p,q)\right| = \frac{2pq}{p+q}\binom{2p-1}{p}\binom{2q-1}{q}$$

	1	2	3	4	5	6
1	1	4	15	56	210	792
2	4	18	72	280	1080	4158
3	15	72	300	1200	4725	18480
4	56	280	1200	4900	19600	77616
5	210	1080	4725	19600	79380	317520
6	792	4158	18480	77616	317520	1280664

In case of $\lambda = 1$ for real Wishart matrix

$$\alpha_{p,q} = 2\left|S_{annNC}(p,q)\right| = 2\,\frac{2pq}{p+q}\binom{2p-1}{p}\binom{2q-1}{q}$$

	1	2	3	4	5	6
1	2	8	30	112	420	1584
2	8	36	144	560	2160	8316
3	30	144	600	2400	9450	36960
4	112	560	2400	9800	39200	155232
5	420	2160	9450	39200	158760	635040
6	1584	8316	36960	155232	635040	2561328

Case of the compound Wishart matrices

The *complex* compound Wishart matrix

compound Wishart matrix

Let $\mathbf{G}_{N,M}$ be an $N \times M$ matrix, the entries X_{ij} of which are independent *complex* centered ($\mathrm{E}(X_{i,j}) = 0$) Gaussian random variables with variance 1 ($\mathrm{Var}(X_{i,j}) = 1$), and let $\left(\mathbf{D}_M\right)_{M \in \mathbb{N}}$ be a sequence of $M \times M$ non-random diagonal matrices for which a limit distribution ρ exists as $M \to \infty$.

Then the $N \times N$ random matrix $\mathbf{X}_N = \frac{1}{N}\mathbf{G}_{N,M}\mathbf{D}_M\mathbf{G}_{N,M}^*$ is called a *complex* <u>*compound Wishat matrix*</u>.

For the sequence of complex compound Wishart matrices $\left(\mathbf{X}_N\right)$, we take the Marchenko-Pastur limit with the asymptotic ratio $M/N \to \lambda$, the limit distribution of which is known as *the compound free Poisson law* $\pi(\rho, \lambda)$.

The moments of the compound free Poisson law

We denote by $m_k(\rho)$ the kth moment of the limit distribution ρ of the sequence $\left(\mathbf{D}_M\right)_{M\in\mathbb{N}}$.

[Voiculescu (1986)]+[Speicher (1994)]

$$\alpha_n = \sum_{\pi\in NC(n)} \prod_{B\in\pi} \lambda\, m_{|B|}(\rho),$$

where $NC(p)$ is the set of the non-crossing partitions of p and $|B|$ stands for the size of a block $B \in \pi$.

Remark

The moment depends not only on the number of blocks but also on the size of each block, which is well-known from the free moment-cumulant formula.

Fluctuations [Mingo, Speicher (2006)]

$$\alpha_{p,q} = \sum_{\pi \in S_{annNC}(p,q)} \prod_{B \in \pi} \lambda\, m_{|C|}(\rho),$$

where $S_{annNC}(p,q)$ is the set of the non-crossing (p,q)-annular permutations and $|C|$ stands for the order of cycle $C \in \pi$.

Remark

The fluctuation formula also depends not only on the number of cycles but also on the order of each cycle similar as one for the moments.

The case for a *real* compound Wishart matrix

The asymptotics and fluctuations for the case of a *real* compound Wishart matrix has been deeply studied by Redelmeier.

It can be found that the first cumulants (the moments) are unchanged, but the values the second cumulant, the covariances, are twice their values in the complex case.

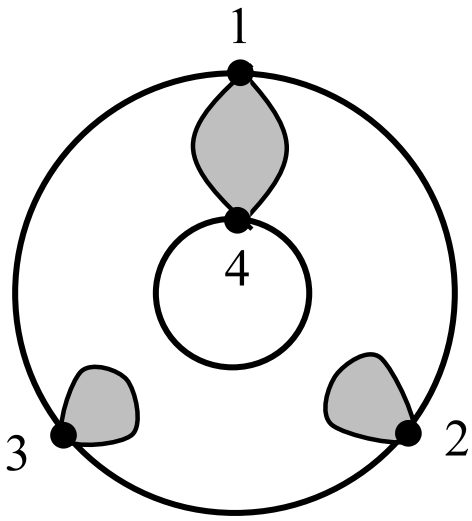
Theorem [Mingo, Speicher (2006)] + [Redelmeier (2011)]

Fluctuations for a *real* compound Wishart matrix:

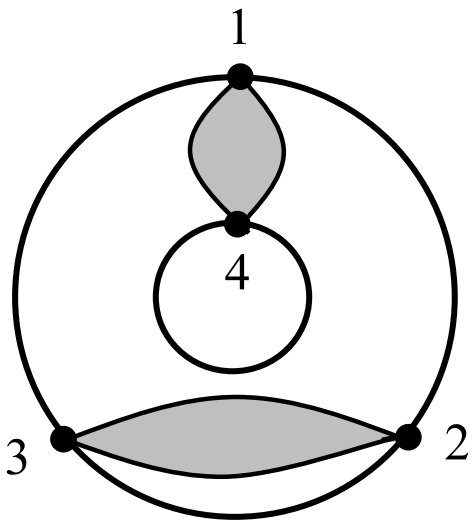
$$\alpha_{p,q} = 2\left(\sum_{\pi \in S_{annNC}(p,q)} \prod_{B \in \pi} \lambda m_{|C|}(\rho)\right).$$

Examples of explicit formulas of fluctuations

$S_{annNC}(3,1)$



$(1,4)(2)(3)$



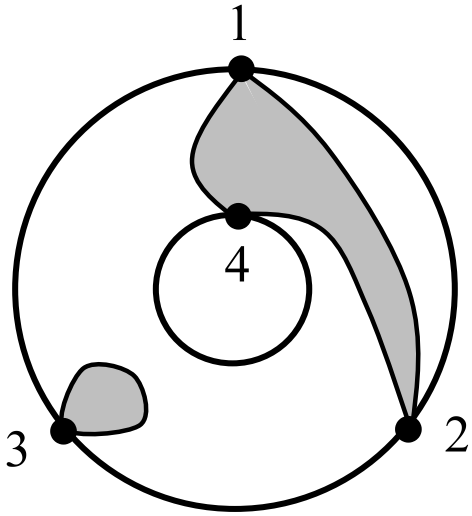
$(1,4)(2,3)$

Same type of permutations:

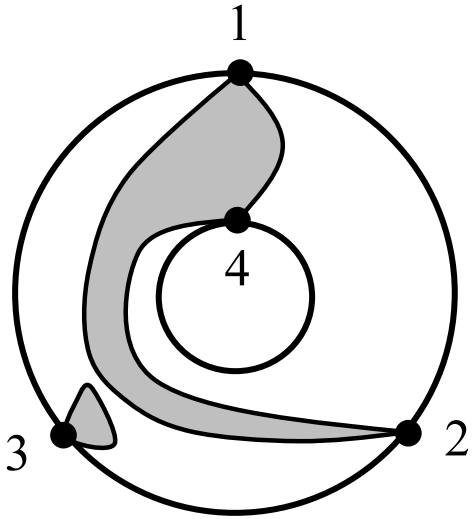
$(2,4)(1)(3)$ $\qquad$ $(2,4)(1,3)$

$(3,4)(1)(2)$ $\qquad$ $(3,4)(1,2)$

$S_{annNC}(3,1)$



$(1,2,4)(3)$



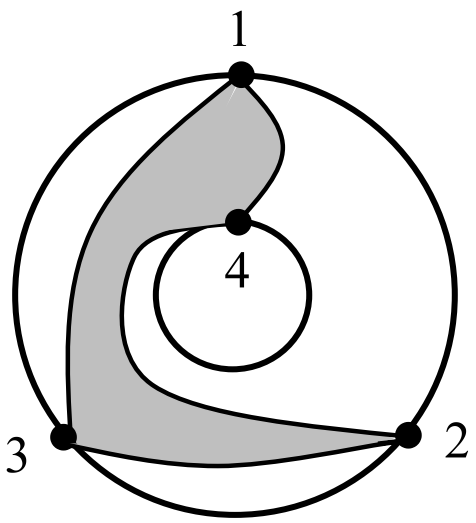
$(1,4,2)(3)$

Same type of permutations:

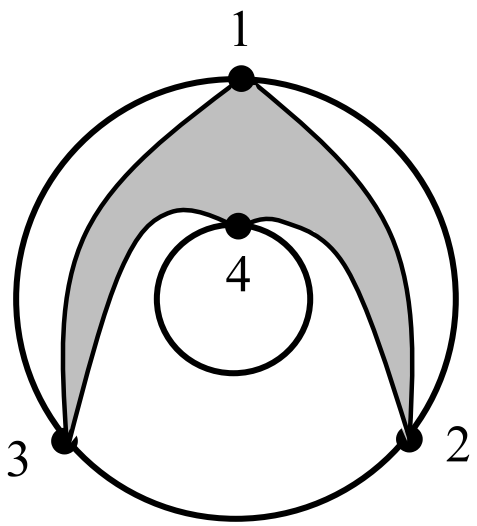
$(2,3,4)(1)$ $\quad$ $(2,4,3)(1)$

$(3,1,4)(2)$ $\quad$ $(3,4,1)(2)$

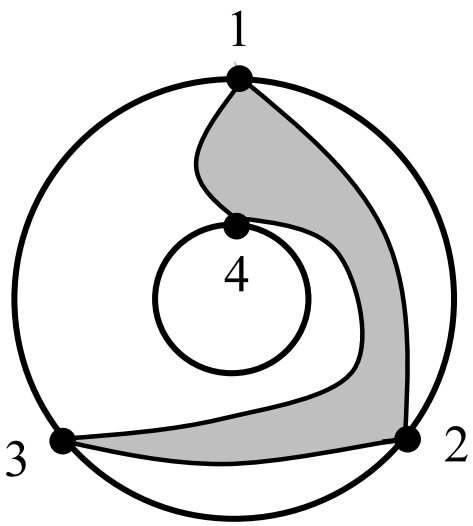
$S_{annNC}(3,1)$



$(1,4,2,3)$



$(2,4,3,1)$



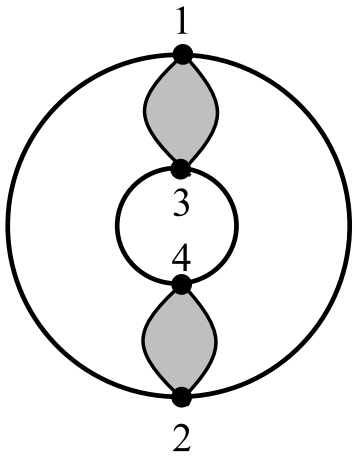
$(3,4,1,2)$

Fluctuation $\alpha_{3,1}$ for a compound Wishart matrix

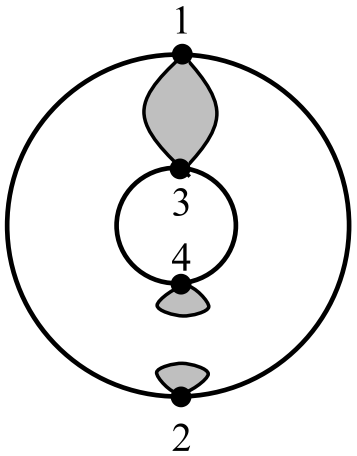
Type	$[1^2\,2^1]$	$[2^2]$	$[1^1\,3^1]$	$[4^1]$
	$(1)(2)(3,4)$	$(1,4)(2,3)$	$(1)(2,3,4)$	$(1,4,2,3)$
	$(2)(3)(1,4)$	$(2,4)(3,1)$	$(1)(2,4,3)$	$(2,4,3,1)$
	$(3)(1)(2,4)$	$(3,4)(1,2)$	$(2)(3,1,4)$	$(3,4,1,2)$
			$(2)(3,4,1)$	
			$(3)(1,2,4)$	
			$(3)(1,4,2)$	
Weight	$\lambda^3 m_1^2 m_2$	$\lambda^2 m_2^2$	$\lambda^2 m_1 m_3$	λm_4

$$\begin{aligned}\alpha_{3,1} &= \lim_{N\to\infty} \mathrm{Cov}\big(\mathrm{Tr}(\mathbf{X}_N^3), \mathrm{Tr}(\mathbf{X}_N)\big) \\ &= 3\lambda^3 m_1^2 m_2 + 3\lambda^2 m_2^2 + 6\lambda^2 m_1 m_3 + 3\lambda m_4\end{aligned}$$

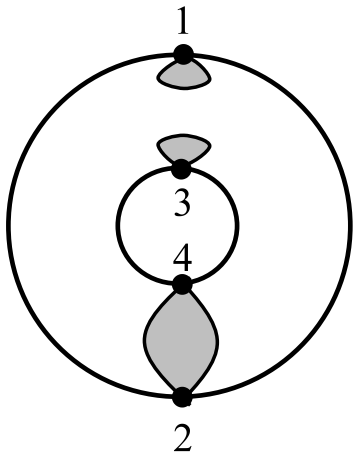
$S_{annNC}(2,2)$



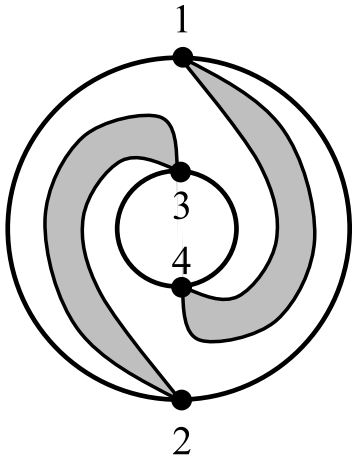
$(1,3)(2,4)$



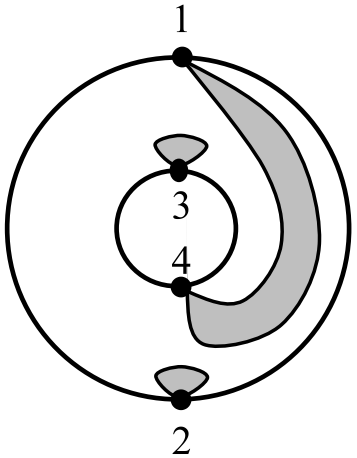
$(1,3)(2)(4)$



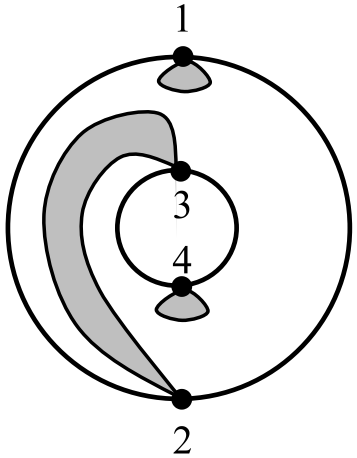
$(2,4)(1)(3)$



$(1,4)(2,3)$

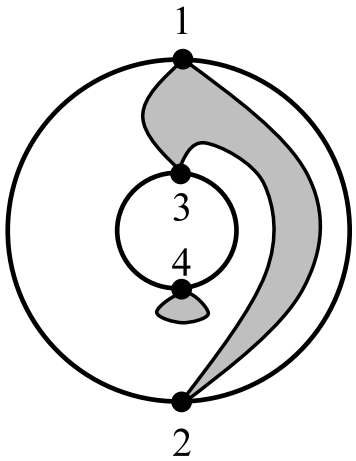


$(1,4)(2)(3)$

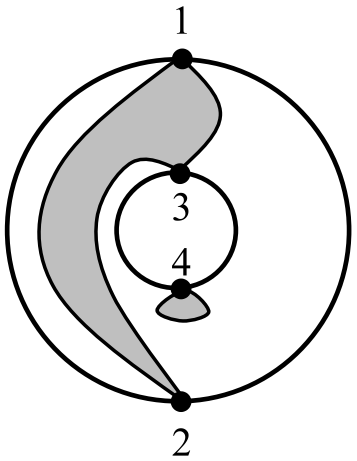


$(2,3)(1)(4)$

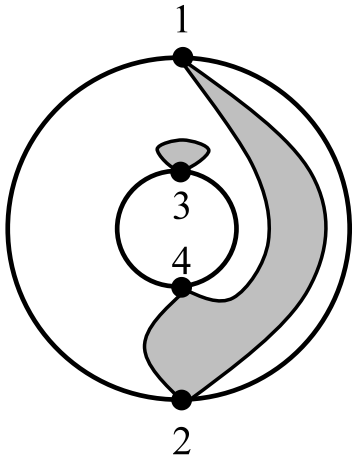
$S_{annNC}(2,2)$



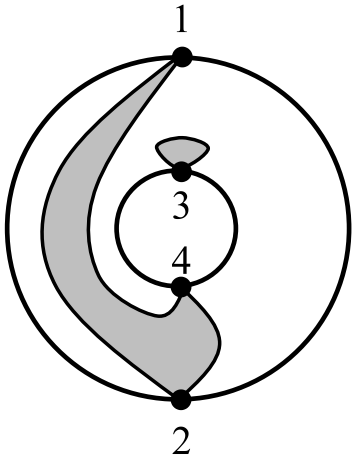
$(1,2,3)(4)$



$(1,3,2)(4)$

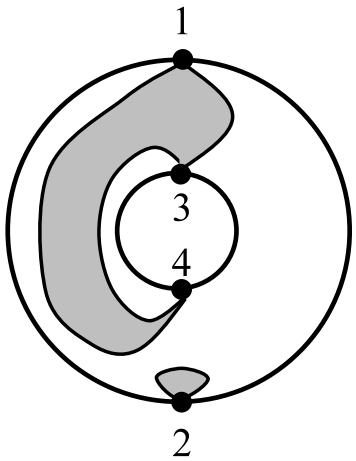


$(1,2,4)(3)$

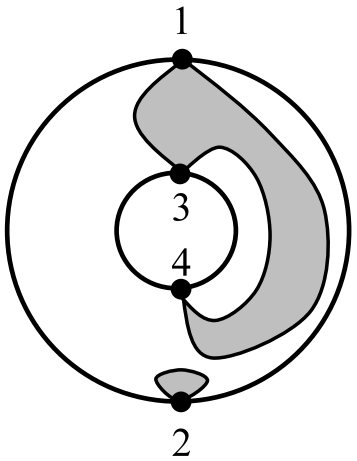


$(1,4,2)(3)$

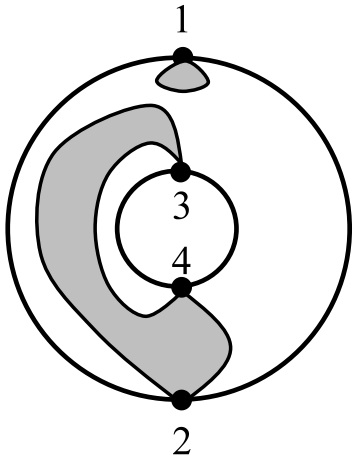
$S_{annNC}(2,2)$



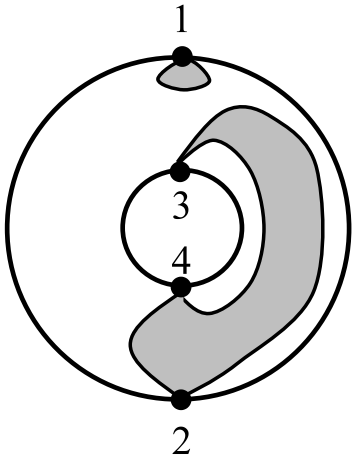
$(1,3,4)(2)$



$(1,4,3)(2)$

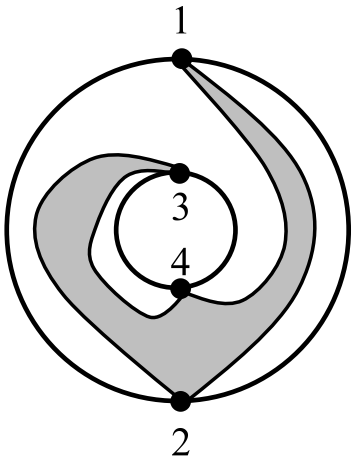


$(2,3,4)(1)$

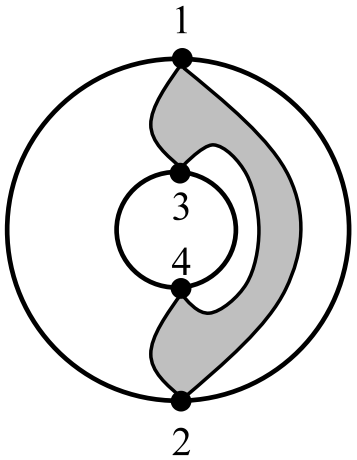


$(2,4,3)(1)$

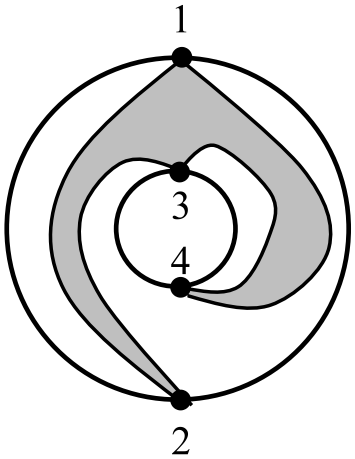
$S_{annNC}(2,2)$



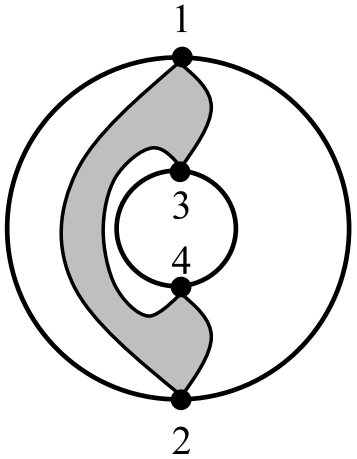
$(1,2,3,4)$



$(1,2,4,3)$



$(1,4,3,2)$



$(1,3,4,2)$

Fluctuation $\alpha_{2,2}$ for a compound Wishart matrix

Type:	$[1^2\,2^1]$	$[2^2]$	$[1^1\,3^1]$	$[4^1]$
	$(1)(3)(2,4)$	$(1,3)(2,4)$	$(1)(2,3,4)$	$(1,2,3,4)$
	$(1)(4)(2,3)$	$(1,4)(2,3)$	$(1)(2,4,3)$	$(1,2,4,3)$
	$(2)(3)(1,4)$		$(2)(1,3,4)$	$(1,3,4,2)$
	$(2)(4)(1,3)$		$(2)(1,4,3)$	$(1,4,3,2)$
			$(3)(1,2,4)$	
			$(3)(1,4,2)$	
			$(4)(1,2,3)$	
			$(4)(1,3,2)$	
Weight:	$\lambda^3 m_1^2 m_2$	$\lambda^2 m_2^2$	$\lambda^2 m_1 m_3$	λm_4

$$\alpha_{2,2} = \lim_{N\to\infty} \mathrm{Cov}\big(\mathrm{Tr}(\mathbf{X}_N^2), \mathrm{Tr}(\mathbf{X}_N^2)\big) = \lim_{N\to\infty} \mathrm{Var}\big(\mathrm{Tr}(\mathbf{X}_N^2)\big)$$
$$= 4\lambda^3 m_1^2 m_2 + 2\lambda^2 m_2^2 + 8\lambda^2 m_1 m_3 + 4\lambda m_4 .$$

Variances for the moments for a compound Wishart matrix

Variances for the first few moments of the limit distribution of a *real* compound Wishart matrix are as follows:

$$
\begin{aligned}
\alpha_{1,1} =& 2\lambda m_2 \\
\alpha_{2,2} =& 2\left(4\lambda^3 m_1^2 m_2 + 2\lambda^2 m_2^2 + 8\lambda^2 m_1 m_3 + 4\lambda m_4\right) \\
\alpha_{3,3} =& 2\Big(9\lambda^5 m_1^4 m_2 + 36\lambda^4 m_1^2 m_2^2 + 36\lambda^4 m_1^3 m_3 + 12\lambda^3 m_2^3 + 72\lambda^3 m_1 m_2 m_3 \\
& + 54\lambda^3 m_1^2 m_4 + 9\lambda^2 m_3^2 + 27\lambda^2 m_2 m_4 + 36\lambda^2 m_1 m_5 + 9\lambda m_6\Big)
\end{aligned}
$$

The functional euqality on the generating functions

Remark.
The functional euqality on the generating functions of the moments and the fluctuations for compound Wishart matrix is known by [Bai and Silverstein (2004)] and also, for more generally, unitary invariant case by [Collins, Mingo, Śniady, and Speicher (2007)].

The generating functions

Let $\{\alpha_n\}_{n\geq 1}$ be the moments of the limit distribution. We write the generating power series $M(x)$ as

$$M(x) = 1 + \sum_{n\geq 1} \alpha_n x^n.$$

Let $\{\alpha_{m,n}\}_{m,n\geq 1}$ be the fluctuation moments (the second order limit). Similar to the first order limit, we define the bivariate generating power series $M(x, y)$ by

$$M(x, y) = \sum_{m,n\geq 1} \alpha_{m,n} x^m y^n.$$

Case of unitary invariant

For a unitary invariant random matrix ensemble (for e.g. GUE or Wishart), the fluctuation moments are totally determined in terms of the (first order) moments by the following relation:

Theorem [Collins, Mingo, Sniady, and Speicher (2007)]

$$M(x,y) = xy\left(\frac{\frac{d}{dx}\big(xM(x)\big)\cdot\frac{d}{dy}\big(yM(y)\big)}{\big(xM(x)-yM(y)\big)^2} - \frac{1}{(x-y)^2}\right).$$

Random matrices with dependent entries

Pfaffel-Schlemm model of dependent entries

Let $\mathbf{X}_{N,M}$ be an $N \times M$ random matrix such that the i th row given by an MA-modeled Gaussian process of the form

$$(X_{i,j}) = \Big(\sum_{\ell=0}^{\infty} c_\ell Z_{i,j-\ell}\Big), \quad c_\ell \in \mathbb{R},$$

where $\{Z_{i,j}\}_{i,j}$ is a family of independent standard Gaussian random variables.

Pfaffel and Schlemm in [PS(2011)] have investigated the Marchenko-Pastur limit of the sample covariance matrix $\mathbf{W}_N$ in the case where dependence in the rows are given by MA model.

Pfaffel and Schlemm's result

In the paper [PS(2011)], the set $\{Z_{i,j}\}$ is given as the family of independent standardized (mean 0 and variance 1) random variables with uniformly bounded fourth moment and the Lindeberg-type condition.

They derived the functional equality for Stiltjes transform $m_\mu(z)$ of the limit spectral measure μ.

Theorem [PS (2011)]

$$\frac{1}{m_\mu(z)} = -z + \lambda \int \frac{x}{1 + x m_\mu(z)} \, d\rho(x),$$

where ρ is the limit spectral measure of the Toeplitz matrix of the form $\big(\gamma(i-j)\big)_{i,j}$ with the autocovariance function $\gamma(h)$.

[HSY(2013)]

In our paper [HSY(2013)],we have understood their result from the point of view in free probability under the assumption of Gaussianity.

By the assumption of Gaussianity, we will lose the generality a little bit, but we will gain many advantages that we can apply the results on the asymptotic analysis of random matrices in free probabolity:

- asymptotic freeness of independent random matrices,
- the second order aysmptotic behavior of the Wishart matrices.

Our model in [HSY(2013)]

Let $\mathbf{X}_{N,M}$ be the $N \times M$ random matrix defined by

$$(X_{i,j}) = \Big(\sum_{\ell=0}^{\infty} c_\ell Z_{i,j-\ell} \Big) \quad c_\ell \in \mathbb{R}$$

where $\{Z_{i,j}\}$ is family of independent standard Gaussian random variables, and let

$$\gamma(h) = \sum_{j=0}^{\infty} c_j c_{j+|h|}$$

be the autocovariance function of the MA-process.

Assume that the spectral measure of the symmetric Toeplitz matrix $\mathbf{\Gamma}_M = \big(\gamma(i-j)\big)_{i,j}^{M}$ converges weakly, as $M \to \infty$, to ρ.

Limit distribution

Theorem [HSY(2013)]

In the Marchenko-Pastur limit $N \to \infty$ and $M/N \to \lambda > 0$, the empirical spectral measure of $\frac{1}{N}\mathbf{X}_{N,M}{}^t\mathbf{X}_{N,M}$ converges almost surely to the compound free Poisson law $\pi(\rho, \lambda)$.

Collorary

In the Marchenko-Pastur limit $N \to \infty$ and $M/N \to \lambda$, the empirical spectral measures of random matrix $\frac{1}{N}\mathbf{X}_{N,M}{}^t\mathbf{X}_{N,M}$ and the compound Wishart matrix $\frac{1}{N}\mathbf{Z}_{N,M}\mathbf{\Gamma}_M{}^t\mathbf{Z}_{N,M}$ converge almost surely to the same compound free Poisson law $\pi(\rho, \lambda)$.

Moments of the limit spectral measure of Toeplitz matrix

The moments of the limit spectral measure ρ of $\mathbf{\Gamma}_M = \big(\gamma(i-j)\big)_{i,j}^M$ can be calculated by Szegö's theorem:

Theorem (Szegö)

Let $f(\omega)$ be the Fourier transform of the autocovariance function $\gamma(h)$,

$$f(\omega) = \sum_{h\in\mathbb{Z}} \gamma(h) e^{-\sqrt{-1}h\omega}.$$

Then for any analytic function F, it holds that

$$\int_{\mathbb{R}} F(t)\rho(dx) = \frac{1}{2\pi}\int_0^{2\pi} F\big(f(\omega)\big)d\omega.$$

Especially, the kth moment $m_k(\rho)$ is given by

$$m_k(\rho) = \frac{1}{2\pi}\int_0^{2\pi} \big(f(\omega)\big)^k d\omega.$$

Remarks on our model

- The assumptions in [PS (2011)] cover only two special cases of the limit spectral measure of ρ with smooth spectral density and ρ with piece-wise constant spectral density. In the case of Gaussians, however, our model of a compound Wishart matrix can relax the assumptions on the spectral density.
- Our model can give random matrix models for *the free Bessel law* of the parameter $s = 2$ and $t = \lambda$. The free Bessel laws have been deeply investigated by [Banica, Belinschi, Capitaine, and Collins], in which they have also given another random matrix models for the free Bessel laws in case of $t = 1$.

Applications to time series analysis

We shall apply fluctuation formula for real Wishart matrices to statistical hypothesis testing for the goodness of the estimated parameters in MA models.

For a given time series data $\{x_j\}_{j=1}^{L}$, we shall approximate it by an MA model,

MA model

$$X_j = c_0 Z_j + c_1 Z_{j-1} + \cdots + c_q Z_{j-q}, \quad q \in \mathbb{N}$$

where $\{Z_j\}$ is the i.i.d. family of standard Gaussian random variables, that is, we regard $\{x_j\}$ as a sample path of the above MA modeled Gaussian process.

Testing statistics

We construct the $N \times M$ matrix $\mathbf{x}$ from given data $\{x_j\}$ as follows:

$$\mathbf{x} = \left(\xi_{i,j}\right) \text{ with } \xi_{i,j} = x_{(i-1)M+j} \text{ where } NM \leq L.$$

Here we note that, in actual situation (in real world problem), we can treat the rows of $\mathbf{x}$ as independent sample paths of an MA model.

We calculate, for instance, the first two sample moments of the covariance matrix $\frac{1}{N}\mathbf{x}^t\mathbf{x}$, that is, $m_k = \frac{1}{N}\mathrm{Tr}\left(\left(\frac{1}{N}\mathbf{x}^t\mathbf{x}\right)^k\right), \quad k = 1, 2.$

We will use these values as the observed values of the testing statistics, the 1st and 2nd moments of the random matrix.

Moments of the limit distribution of Toeplitz matrix

As we have mentioned, the 1st and 2nd moments are Gussian distributed and the means and the variances depend only on the MA parameters $\{c_i\}$ and the ratio $\lambda = M/N$.

The moments $m_k(\rho)$ of the limit distribution ρ of the Toeplitz matrix $\Gamma = \big(\gamma(i-j)\big)$ can be given via the Fourier transform of the autocovariance function $\gamma(h) = \sum_{j=0}^{q} c_j c_{j+|h|}$.

Gaussian distributed testing statistics

We denote $m_k(\rho)$ by simply m_k.

Testing statistic, the 1st momemt Gaussian distributed

mean: λm_1

variance: $2(\lambda m_2)/N^2$

Testing statistic, the 2nd momemt Gaussian distributed

mean: $\lambda^2 m_1^2 + \lambda m_2$

variance: $2\big(4\lambda^3 m_1^2 m_2 + 2\lambda^2 m_2^2 + 8\lambda^2 m_1 m_3 + 4\lambda m_4\big)/N^2$

For the 2nd momemt of the test statistics, we need upto the 4th moments of the limit distribution ρ.

Numerical simulation

We consider the MA model of degree 2,

$$X_j = 1.0Z_j + 1.0Z_{j-1} + 1.0Z_{j-2}.$$

As the MA parameters are

$$c_0 = 1,\ c_1 = 1,\ c_2 = 1,\ \text{ and }\ c_k = 0\ (k \geq 3),$$

the autocovariance function $\gamma(h)$ is given by

$$\gamma_c(0) = 3,\ \gamma_c(1) = 2,\ \gamma_c(2) = 1,\ \text{ and }\ \gamma_c(k) = 0\,(k \geq 3).$$

Thus its Fourier transform becomes

$$f(\omega) = 3 + 2\big(2\cos\omega + \cos 2\omega\big),$$

which gives the first four moments of the limit measure ρ as

$$m_1(\rho) = 3,\ m_2(\rho) = 19,\ m_3(\rho) = 141,\ \text{ and }\ m_4(\rho) = 1107.$$

Case of $M = 50$ and $N = 50$

As the ratio is $\lambda = 1.0$, the moment and the fluctuation for the second moment of a real compound Wishart matrix are given by $\alpha_2 = 28$ and $\alpha_{2,2} = 18436$, respectively.

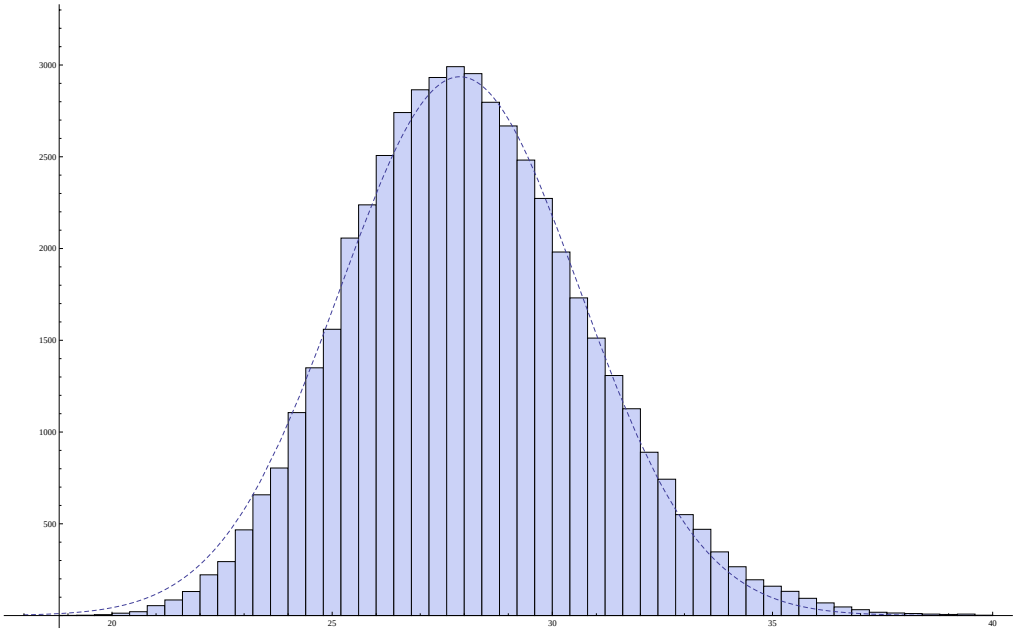
Since this is the case of $N = 50$, the second moment μ_2 could be observed according to the normal distribution

$$\mathcal{N}\big(\alpha_2,\, \alpha_{2.2}/N^2\big) = \mathcal{N}\big(28,\, 7.3852\big).$$

Here we shall give the results of the numerical simulation on the second moment μ_2 for 50000 samples of 50×50 data matrices, the mean and the variance of which have been obtained as 28.1163 and 7.3327, respectively.

Case of $M = 50$ and $N = 50$

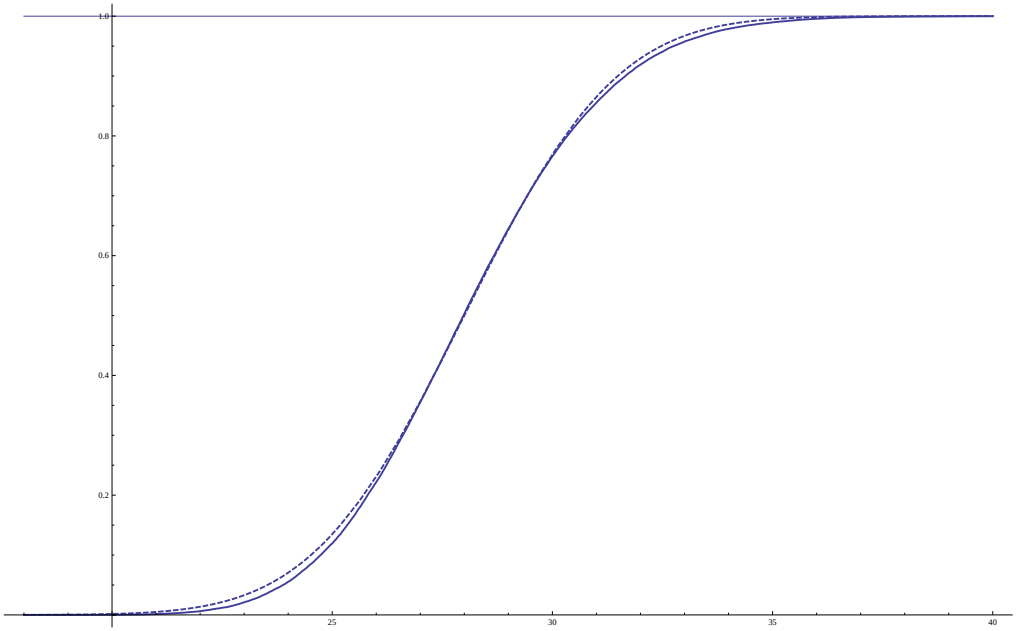
The histogram of 50000 observed data with the dashed line for the scaled normal curve of $\mathcal{N}(28, 7.3852)$.



Histogram of the second moments of 50000 samples of 50×50

Case of $M = 50$ and $N = 50$

The cumulative distribution function of 50000 observed data, where the dashed line indicates the cumulative distribution function of the Gaussian distribution $\mathcal{N}(28, 7.3852)$.



Cumulative distribution function of 50000 samples of 50×50

Case of $M = 30$ and $N = 30$

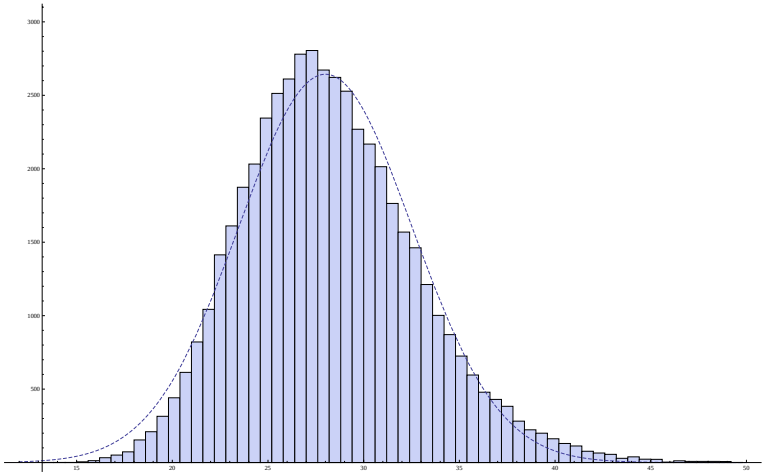
In this case, the ratio remains $\lambda = 1.0$ and the moment and fluctuation are unchanged as $\alpha_2 = 28$ and $\alpha_{2,2} = 18436$.
Since this is the case of $N = 30$, the second moment μ_2 could be observed according to the normal distribution

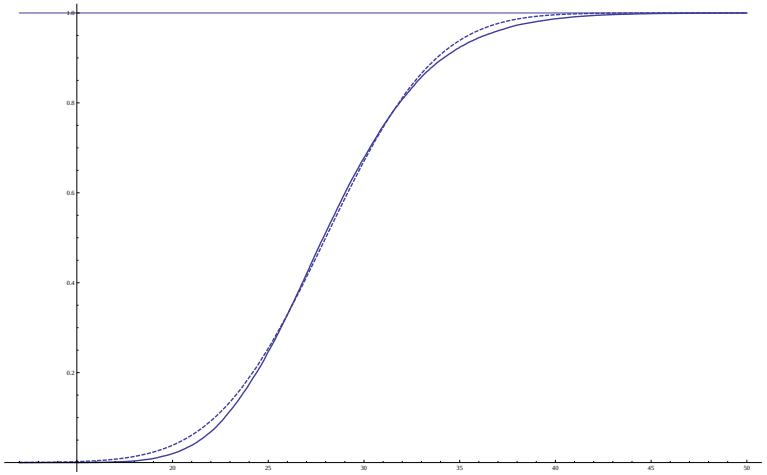
$$\mathcal{N}\big(\alpha_2,\, \alpha_{2.2}/N^2\big) = \mathcal{N}\big(28,\, 20.48444\big).$$

The mean and the variance of 50000 samples of the numerically simulated matrices of the size 30×30 are 28.2428 and 20.7449, respectively.

Case of $M = 30$ and $N = 30$

The histogram and the cumulative distribution function of 50000 observed data are irustrated below. the dashed lines stand for the Gaussian distribution $\mathcal{N}(28, 20.48444)$.





Case of $M = 60$ and $N = 40$

The ratio is given as $\lambda = 3/2$, thus the moment and fluctuation become $\alpha_2 = 195/4 = 48.75$ and $\alpha_{2,2} = 36378$, respectively.

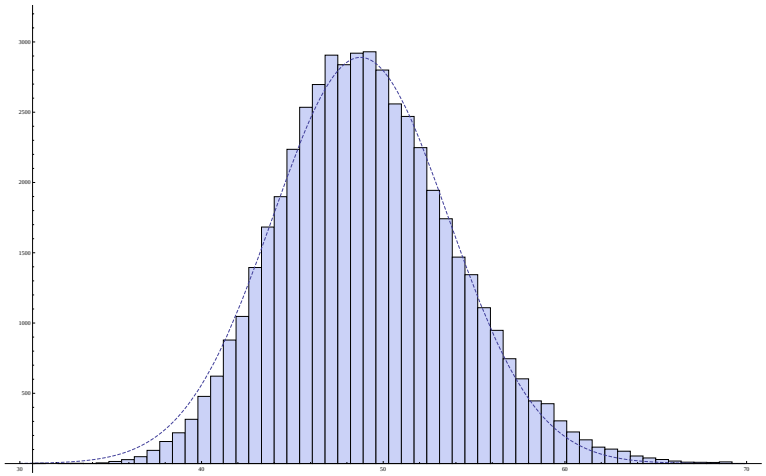
Since this is the case of $N = 40$, the second moment μ_2 could be observed according to the normal distribution

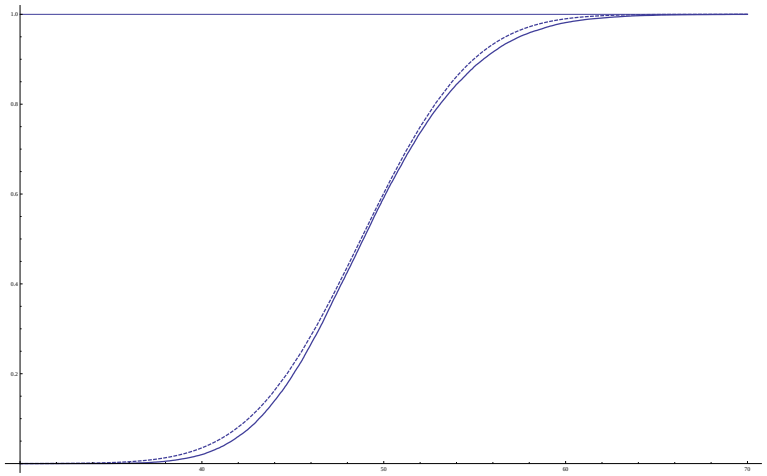
$$\mathcal{N}\left(\mu_2, \alpha_{2.2}/N^2\right) = \mathcal{N}\left(48.75, 22.7363\right).$$

The mean and the variance of 50000 samples of the numerically simulated matrices of the size 40×60 are 49.1418 and 23.3413, respectively.

Case of $M = 60$ and $N = 40$

The histogram and the cumulative distribution function of them are irustrated below, where the dashed lines stand for the Gaussian distribution $\mathcal{N}(28, 20.48444)$.





Statistical hypothesis testing

According to the above numerical observations, we will propose the statistical hypothesis testing on time series models.

(1) Let the time series data $\{x_j\}_{j=1}^{L}$ be given, and consider a certain time series model, for instance, an MA model of finite degree.

(2) Here the null hypothesis H_0 is that *given time series data could be generated by the model that we have assumed*.

(3) We construct $N \times M$ matrix $\mathbf{x}$ from the given data as before, and compute the second sample moment $\mu_2 = \dfrac{1}{N}\mathrm{Tr}\big((\frac{1}{N}\mathbf{x}^t\mathbf{x})^2\big)$.

(4) Using the parameters of our assumed model, we estimate the mean and the variance of the Gaussian distribution to which the second moment should be distributed under the above null hypothesis.

(5) Then we shall apply the Z-test of two-tailed to see the statistical significance.

Remarks on our proposal

- The above statistical testing can be applicable to
  <u>not only MA but also AR and more general ARMA</u>
  because the expected moments and fluctuations can be calculated depending only on the autocovariance function γ.

- While the moments m_k of the weak limit ρ of the symmetric Toeplitz matrix $\big(\gamma(i-j)\big)$ are needed to estimate the mean and the variance of the normal distribution for the Z-test, the explicit form of the autocovariance function is not exactly needed but its Fourier transform is essential.

Fourier transform of the autocovariance function

On the Fourier transform

For ARMA(p, q) model

$$d_0 X_j + d_1 X_{j-1} + \cdots + d_p X_{j-p} = c_0 Z_j + c_1 Z_{j-1} + \cdots + c_q Z_{j-q},$$

the autocovariance function does not in general have a simple closed form, but its Fourier transform is simply obtained as

$$f(\omega) = \left| \frac{Q\left(e^{\sqrt{-1}\omega}\right)}{P\left(e^{\sqrt{-1}\omega}\right)} \right|^2, \quad \omega \in [0, 2\pi),$$

where P and Q are polynomials respectively given by

$$P(z) = d_0 + d_1 z + \cdots + d_p z^p \text{ and } Q(z) = c_0 + c_1 z + \cdots + c_q z^q.$$

Numerical example

We consider the MA model

$$X_j = 1.0Z_j + 0.3Z_{j-1} - 0.5Z_{j-2},$$

and generate the simulated time series data of length 2500, and constructed $N \times N = 50 \times 50$ data matrix $\mathbf{x}$.
Then the second sample moments $\frac{1}{N}\mathrm{Tr}\big((\frac{1}{N}\mathbf{x}^t\mathbf{x})^2\big)$ has been obtained as $\widetilde{\mu_2} = 4.01996$.

Numerical example

We apply the fitting of time series models to our generated data by *Mathematica*.

Mathematica has selected ARMA(1,2) with the parameters

AR Coef.		MA Coef.	
$d_1 =$	-0.032794	$c_1 =$	0.305035
		$c_2 =$	-0.477926

$v = 1.012043$ (variance of white noise)

as an optimal model by AIC.

Numerical example

The expected 2nd moment and the fluctuation can be calculated from the parameters as follows: The Fourier transform of the autocovariance function $f(\omega)$ is given as

$$f(\omega) = \left| \frac{1.0 + 0.305035\, e^{w\sqrt{-1}} - 0.477926\, e^{2w\sqrt{-1}}}{1. + 0.032794\, e^{w\sqrt{-1}}} \right|^2,$$

by calculating $m_k = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} \left(f(\omega)\right)^k d\omega$ for $k = 1, 2, 3, 4$, we have

$$\mu_2 = (m_1^2 + m_2)v^2 = 4.03544,$$
$$\sigma^2 = 2(4m_1^2 m_2 + 2m_2^2 + 8m_1 m_3 + 4m_4)v^4/50^2 = 0.0853451$$

because the ratio $\lambda = 1.0$.

Numerical example

Thus we obtain z-value as

$$z = \frac{\widetilde{\mu}_2 - \mu_2}{\sigma} = \frac{4.01996 - 4.03544}{\sqrt{0.085345}} = -0.05298.$$

The AIC for this model is calculated as 39.92854.

We will list some candidates (models) and its calculated z-value and AIC for each model.

Numerical example

Model	Abs. of z	AIC
ARMA (1,2)	0.05298	39.92854
ARMA (2,2)	0.06713	41.98385
ARMA (2,1)	1.48927	116.96120
MA(2)	3.00499	165.13917

The goodness of models could be selected by z-values (at least in our example).

AIC of ARMA model

$$AIC = N \log \widehat{\sigma_\varepsilon}^2 + 2(p+q+1)$$

where $\widehat{\sigma_\varepsilon}^2$ is the variance of prediction errors, which is given by

$$\widehat{\sigma_\varepsilon}^2 = \frac{1}{N-p} \sum_{n=p+1}^{N} \left(\sum_{i=0}^{p} d_i \, x_{n-i} \right)^2 \Big/ \left(\sum_{j=0}^{q} c_j^2 \right)$$

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Diagonally deformed GUE & GOE

Diagonally deformed GUE

We treat the following Wigner matrix of Gaussian entries:

$\mathbf{X}_n = (X_{ij})$ is an $n \times n$ Wigner *complex hermitian* matrix, in which X_{ii} $(1 \le i \le n)$ are i.i.d. real centered Gaussian random variables of variance σ^2, and X_{ij} $(1 \le i < j \le n)$ are i.i.d. complex centered Gaussian random variables of variance η^2.

We write such a Winger complex hermitian matrix of Gaussian entries as symbolically

$$\begin{pmatrix} \sigma^2 & & \eta^2 \\ & \ddots & \\ \eta^2 & & \sigma^2 \end{pmatrix} \quad \text{or, more simply,} \quad \mathrm{Wig}_{\mathbb{C}}(\sigma^2; \eta^2)_n.$$

Diagonally deformed GUE

$\mathrm{Wig}_{\mathbb{C}}(\sigma^2;\eta^2)_n$ is not unitary invariant in general, but invariant only when $\mathrm{Wig}_{\mathbb{C}}(\eta^2;\eta^2)_n$, which corresponds to the scaled GUE.
We denote this scaled GUE by $\mathrm{GUE}(\eta^2)_n$.

We consider the random matrix ensemble $\left(\frac{1}{\sqrt{n}}\mathrm{Wig}_{\mathbb{C}}(\sigma^2;\eta^2)_n\right)$ and see the fluctuation moments, where the limit eigenvalue distribution of which is, of course, the centered semicircle law of variance η^2.

Diagonally deformed GOE

$\mathbf{X}_n = (X_{ij})$ is an $n \times n$ Wigner *real symmetric* matrix of Gaussian entries, in which X_{ii} $(1 \leq i \leq n)$ are i.i.d. real centered Gaussian random variables of varinace σ^2, and X_{ij} $(1 \leq i < j \leq n)$ are i.i.d. real centered Gaussian random variables of varinace η^2.

We write such a Winger real symmetric matrix as symbolically

$$\begin{pmatrix} \sigma^2 & & \eta^2 \\ & \ddots & \\ \eta^2 & & \sigma^2 \end{pmatrix}$$

or, more simply, $\mathrm{Wig}_{\mathbb{R}}(\sigma^2; \eta^2)_n$.

Diagonally deformed GOE

$\mathrm{Wig}_{\mathbb{R}}(\sigma^2;\eta^2)_n$ is not orthogonal invariant in general, but invariant only when $\mathrm{Wig}_{\mathbb{R}}(2\eta^2;\eta^2)_n$, which corresponds to the scaled GOE. We denote this scaled GOE by $\mathrm{GOE}(\eta^2)_n$.

Similar to complex case, we consider the random matrix ensemble $\left(\frac{1}{\sqrt{n}}\mathrm{Wig}_{\mathbb{R}}(\sigma^2;\eta^2)_n\right)$ and find the fluctuation formula, where the limit eigenvalue distribution of which is, of course, given by the centered semicircle law of variance η^2.

The second order free cumulants and asymptotic freeness

The first order free cumulants

Let $\{\alpha_n\}_{n\geq 1}$ be the moments of the limit distribution. We write the generating power series $M(x)$ as

$$M(x) = 1 + \sum_{n\geq 1} \alpha_n x^n.$$

The (first order) free cumulants $\{\kappa_n\}_{n\geq 1}$ is defined by the functional equation

$$C\big(xM(x)\big) = M(x),$$

where we write the generating power series $C(x)$ as

$$C(x) = 1 + \sum_{n\geq 1} \kappa_n x^n,$$

which is essentially the same as Voiculescu's R-transform.

The second order free cumulants

Let $\left\{\alpha_{m,n}\right\}_{m,n\geq 1}$ be the fluctuation moments (the second order limit). Similar to the first order limit, we define the bivariate generating power series $M(x,y)$ by

$$M(x,y) = \sum_{m,n\geq 1} \alpha_{m,n} x^m y^n.$$

Then the corresponding (the second order) free cumulants $\left\{\kappa_{m,n}\right\}_{n\geq 1}$ are defined as follows:
We put the bivariate formal power series $C(x,y)$ by

$$C(x,y) = \sum_{m,n\geq 1} \kappa_{m,n} x^m y^n.$$

The second order free cumulants

Then it is required the following functional relations for the second order free cumulants:

$$M(x,y) = C\big(xM(x), yM(y)\big) \cdot \frac{\frac{d}{dx}\big(xM(x)\big)}{M(x)} \cdot \frac{\frac{d}{dy}\big(yM(y)\big)}{M(y)} + xy\left(\frac{\frac{d}{dx}\big(xM(x)\big) \cdot \frac{d}{dy}\big(yM(y)\big)}{\big(xM(x) - yM(y)\big)^2} - \frac{1}{(x-y)^2}\right).$$

The name of the second order free cumulants

Theorem [CMSS]

Let $\mathcal{A}$ and $\mathcal{B}$ be two random matrix ensembles which are asymptotically free. Then we have for all $m, n \geq 1$ that

$$\kappa_n^{\mathcal{A}+\mathcal{B}} = \kappa_n^{\mathcal{A}} + \kappa_n^{\mathcal{B}} \quad \text{and} \quad \kappa_{m,n}^{\mathcal{A}+\mathcal{B}} = \kappa_{m,n}^{\mathcal{A}} + \kappa_{m,n}^{\mathcal{B}}.$$

The above Theorem gives the reason why the $\kappa_{m,n}$ are called the second order *free cumulants* because they linearize the problem in calculating the fluctuations of the sum of asymptotically free random matrices.

Definition (Asymptotic freeness)

Two random matrix ensembles $\mathcal{A} = \left(\mathbf{A}_N\right)_N$ and $\mathcal{B} = \left(\mathbf{B}_N\right)_N$ with limit eigenvalue distributions are asymptotically free if it follows that for all $p \geq 1$ and all $n(1), m(1), \ldots, n(p), m(p) \geq 1$

$$\lim_{N\to\infty} E\Big[\mathrm{tr}\Big(\big(\mathbf{A}_N^{n(1)} - \alpha_{n(1)}^{\mathcal{A}}\mathbf{1}\big)\cdot\big(\mathbf{B}_N^{m(1)} - \alpha_{m(1)}^{\mathcal{B}}\mathbf{1}\big)\cdots \\ \cdots\big(\mathbf{A}_N^{n(p)} - \alpha_{n(p)}^{\mathcal{A}}\mathbf{1}\big)\cdot\big(\mathbf{B}_N^{m(p)} - \alpha_{m(p)}^{\mathcal{B}}\mathbf{1}\big)\Big)\Big] = 0.$$

Asymptotic freeness gives the rule to calculate all mixed moments that is, expression

$$\lim_{N\to\infty} E\Big[\mathrm{tr}\Big(\mathbf{A}_N^{n(1)}\mathbf{B}_N^{m(1)}\cdots\mathbf{A}_N^{n(p)}\mathbf{B}_N^{m(p)}\Big)\Big]$$

can be determined by the limit moments of $\mathcal{A}$ and $\mathcal{B}$, uniquely.

Examples of asymptotically free ensembles

Voiculescu

- Independent sequences of GUE are asymptoticaly free.
- GUE and diagonal constant matrices are asymptoticaly free.
- For Haar unirary U, $\{U, U^*\}$ and diagonal constant matrix D are asymptoticaly free.

Since any unitary invariant matrix can be written as UDU^* where D is a diagonal matrix and U is Haar unitary matrix, thus we have:

Let A_N and B_N be $N \times N$ independent random matrices such that each of them has the limit distribution, and at least one of them is unitary invariant. Then (A_N) and (B_N) are asymptotically free.

More examples

Dykema - Speicher

Let X_N be a selfadjoint Wigner matrix, such that the distribution of the entries is centered and has all moments, and let A_N be a random matrix which is independent from X_N. If A_N has the limit eigenvalue distribution and uniformly bounded in operator norm for N then A_N and X_N are asymptotically free.

The above matrices are not unitary invariant in general, but they are asymptotically free.

For unitary invariant random matrices

The following Proposition plays important role:

Proposition

The second order free cumulants for a unitary invariant random matrix ensemble (especially, for GUE) vanish identically $C(x,y)=0$.

In this case, hence, the fluctuation moments are totally determined in terms of the (first order) moments.

$$M(x,y) = xy\left(\frac{\frac{d}{dx}\big(xM(x)\big)\cdot\frac{d}{dy}\big(yM(y)\big)}{\big(xM(x)-yM(y)\big)^2} - \frac{1}{(x-y)^2}\right).$$

Random diagonal matrices

We will see a trivial but an important fact on the second order free cumulants for the random diagonal matrices.

Lemma

Let $\mathbf{D}_n = diag(D_1, D_2, \ldots, D_n)$ *be the random diagonal matrix, where* D_i*'s are i.i.d. real centered Gaussian of variance* ζ^2*, and consider the sequence* $\mathcal{D} = \left(\frac{1}{\sqrt{N}}\mathbf{D}_N\right)$.

Then the first order limits α_p *for* $\mathcal{D}$ *exist and converge to* 0 *for all* $p \geq 1$*, and hence, all the first order free cumulants wil vanish.*

Futhermore, it is easy to find that the fluctuatin moments $\alpha_{p,q}$ *also exist for all* $p, q \geq 1$ *and vanish other than* $\alpha_{1,1} = \zeta^2$*. Thus it follows that only the second free cumulant* $\kappa_{1,1}$ *has non-zero value* $\kappa_{1,1} = \zeta^2$*, that is,*

$$C(x, y) = \zeta^2 xy.$$

Fluctuations of the Wigner matrices

Decomposition of $\mathrm{Wig}_{\mathbb{C}}(\sigma^2;\eta^2)_n$ into the independent sum.

<u>Case of $\sigma^2 \geq \eta^2$</u>

$$\begin{pmatrix} \sigma^2 & & \eta^2 \\ & \ddots & \\ \eta^2 & & \sigma^2 \end{pmatrix} = \begin{pmatrix} \eta^2 & & \eta^2 \\ & \ddots & \\ \eta^2 & & \eta^2 \end{pmatrix} + \begin{pmatrix} (\sigma^2-\eta^2) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & (\sigma^2-\eta^2) \end{pmatrix}$$

$$\mathrm{Wig}_{\mathbb{C}}(\sigma^2;\eta^2)_n = \mathrm{GUE}(\eta^2)_n + \mathrm{Dia}(\sigma^2-\eta^2)_n$$

$\mathrm{GUE}(\eta^2)$ is the scaled GUE matrix thus unitary invariant.

Fluctuation formula

The limit eigenvalue distribution of $\mathcal{W} = \left(\frac{1}{\sqrt{n}}\mathrm{Wig}_{\mathbb{C}}(\sigma^2;\eta^2)_n\right)$ is the same as one of $\left(\frac{1}{\sqrt{n}}\mathrm{GUE}(\eta^2)_n\right)$, that is, the centered semicircle law of variance η^2.

Hence the moment generating function $M(z)$ for $\mathcal{W}$ is given by

$$M(z) = 1 + \sum_{k\geq 1} C_k \eta^{2k} z^{2k} = \sum_{k\geq 0} \frac{1}{k+1}\binom{2k}{k}\eta^{2k} z^{2k},$$

where C_k is the kth Catalan number.

Fluctuation formula

The second order free cumulants of $\mathcal{W} = \left(\mathrm{Wig}_{\mathbb{C}}(\sigma^2;\eta^2)_n\right)$ should be given by those for $\left(\mathrm{Dia}(\sigma^2-\eta^2)_n\right)$. Namely,

$$\kappa_{1,1} = \sigma^2 - \eta^2, \quad \kappa_{p,q} = 0 \text{ for } (p,q) \neq (1,1).$$

Hence the bivariate generating function of the second order free cumulants $C(x,y)$ for $\mathcal{W}$ is given by

$$C(x,y) = (\sigma^2 - \eta^2)\, xy.$$

Relation of the fluctuation moments

Using the functional relation of the fluctuation moments, we have

$$
\begin{aligned}
M(x,y) =& C\big(xM(x), yM(y)\big) \cdot \frac{\frac{d}{dx}\big(xM(x)\big)}{M(x)} \cdot \frac{\frac{d}{dy}\big(yM(y)\big)}{M(y)} \\
&+ xy\left(\frac{\frac{d}{dx}\big(xM(x)\big) \cdot \frac{d}{dy}\big(yM(y)\big)}{\big(xM(x)-yM(y)\big)^2} - \frac{1}{(x-y)^2}\right) \\
=& \underbrace{(\sigma^2-\eta^2)\, xy \frac{d}{dx}\big(xM(x)\big) \cdot \frac{d}{dy}\big(yM(y)\big)}_{\text{Part of the 2nd order free cumulant}} \\
&+ \underbrace{xy\left(\frac{\frac{d}{dx}\big(xM(x)\big) \cdot \frac{d}{dy}\big(yM(y)\big)}{\big(xM(x)-yM(y)\big)^2} - \frac{1}{(x-y)^2}\right)}_{\text{This part is exactly the same as for GUE}(\eta^2)}
\end{aligned}
$$

Part of the 2nd order free cumulant

$$
\begin{aligned}
&(\sigma^2-\eta^2)\left\{xy\frac{d}{dx}(xM(x))\cdot\frac{d}{dy}(yM(y))\right\}\\
=&(\sigma^2-\eta^2)\left(\sum_{k\geq 0}\frac{2k+1}{k+1}\binom{2k}{k}\eta^{2k}x^{2k+1}\right)\left(\sum_{\ell\geq 0}\frac{2\ell+1}{\ell+1}\binom{2\ell}{\ell}\eta^{2\ell}y^{2\ell+1}\right)\\
=&(\sigma^2-\eta^2)\sum_{k,\ell\geq 0}\binom{2k+1}{k}\binom{2\ell+1}{\ell}\eta^{2k+2\ell}x^{2k+1}y^{2\ell+1}.
\end{aligned}
$$

$$
\begin{aligned}
&[x^py^q](\sigma^2-\eta^2)\left\{xy\frac{d}{dx}(xM(x))\cdot\frac{d}{dy}(yM(y))\right\}\\
&=\begin{cases}\binom{p}{\frac{p-1}{2}}\binom{q}{\frac{q-1}{2}}(\sigma^2-\eta^2)\,\eta^{p+q-2} & \text{if } p \text{ and } q \text{ are odd,}\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}
$$

Part of GUE

The latter part of the generating function is the same as for GUE(η^2), thus we have

$$[x^p y^q]\, xy \left\{ \frac{\frac{d}{dx}\big(xM(x)\big) \cdot \frac{d}{dy}\big(yM(y)\big)}{\big(xM(x) - yM(y)\big)^2} - \frac{1}{(x-y)^2} \right\}$$

$$= \begin{cases} \dfrac{pq}{2(p+q)} \overset{p}{\binom{p}{\frac{p}{2}}} \overset{q}{\binom{q}{\frac{q}{2}}} \eta^{p+q} & \text{if } p \text{ and } q \text{ are even,} \\ \dfrac{2pq}{p+q} \overset{p-1}{\binom{p-1}{\frac{p-1}{2}}} \overset{q-1}{\binom{q-1}{\frac{q-1}{2}}} \eta^{p+q} & \text{if } p \text{ and } q \text{ are odd,} \\ 0 & \text{ortherwise.} \end{cases}$$

Fluctuation formula of Wigner hermitian matrix

The part of the 2nd order free cumulant will contribute only the case when p, q are odd. Thus we have the following formula:

In case the of $\sigma^2 \geq \eta^2$, the fluctuation moment $\alpha_{p,q}$ is given as

- if p and q are even:
  $$\frac{pq}{2(p+q)} \binom{p}{\frac{p}{2}} \binom{q}{\frac{q}{2}} \eta^{p+q},$$
- if p and q are odd:
  $$\frac{2pq}{p+q} \binom{p-1}{\frac{p-1}{2}} \binom{q-1}{\frac{q-1}{2}} \eta^{p+q} + \binom{p}{\frac{p-1}{2}} \binom{q}{\frac{q-1}{2}} \left(\frac{\sigma^2}{\eta^2} - 1\right) \eta^{p+q}$$
- ortherwise:
  0.

Case of $\sigma^2 < \eta^2$

$$\begin{pmatrix} \sigma^2 & & \eta^2 \\ & \ddots & \\ \eta^2 & & \sigma^2 \end{pmatrix} = \begin{pmatrix} \sigma^2 & & \sigma^2 \\ & \ddots & \\ \sigma^2 & & \sigma^2 \end{pmatrix} + \begin{pmatrix} 0 & & \eta^2 - \sigma^2 \\ & \ddots & \\ \eta^2 - \sigma^2 & & 0 \end{pmatrix}$$

$$\mathrm{Wig}_{\mathbb{C}}(\sigma^2; \eta^2)_n = \mathrm{GUE}(\sigma^2)_n + \mathrm{Wig}_{\mathbb{C}}(0; \eta^2 - \sigma^2)_n$$

The second free cumulants of $\mathrm{Wig}_{\mathbb{C}}(0;\eta^2-\sigma^2)_n$

The limit eigenvalue distribution of $\mathcal{E}_1 = \left(\frac{1}{\sqrt{n}}\mathrm{Wig}_{\mathbb{C}}(0;\eta^2-\sigma^2)_n\right)$ is the centered semicircle law of variance $(\eta^2-\sigma^2)$. Thus we have on the first order free cumulant that :

$$\kappa_2(\mathcal{E}_1) = \eta^2-\sigma^2, \quad \kappa_n(\mathcal{E}_1) = 0 \ \text{ for } n \neq 2.$$

Concerning the second order free cumulants, it follows that

$$\kappa_{1,1}(\mathcal{E}_1) = -(\eta^2-\sigma^2), \quad \kappa_{p,q}(\mathcal{E}_1) = 0 \ \text{ for } \ (p,q) \neq (1,1).$$

The reasons for this result are in the next slide.

Asymptotic freeness of Wigner matrices

On the second order free cumulants, note the following facts:

- $\mathrm{Wig}_{\mathbb{C}}(0;\eta^2-\sigma^2)_n + \mathrm{Dia}(\eta^2-\sigma^2)_n = \mathrm{GUE}(\eta^2-\sigma^2)_n$
- $\mathrm{GUE}(\eta^2-\sigma^2)_n$ is unitary invariant, thus all the second order free cumulants will vanish
- In the second order free cumulants of $\left(\frac{1}{\sqrt{n}}\mathrm{Dia}(\eta^2-\sigma^2)_n\right)$, only $\kappa_{1,1}$ has non-zero value $\kappa_{1,1}=\eta^2-\sigma^2$.
- $\left(\frac{1}{\sqrt{n}}\mathrm{Wig}_{\mathbb{C}}(0;\eta^2-\sigma^2)_n\right)$ and $\left(\frac{1}{\sqrt{n}}\mathrm{Dia}(\eta^2-\sigma^2)_n\right)$ are asymptotically free.

Both of $\left(\frac{1}{\sqrt{n}}\mathrm{Wig}_{\mathbb{C}}(0;\eta^2-\sigma^2)_n\right)$ and $\left(\frac{1}{\sqrt{n}}\mathrm{Dia}(\eta^2-\sigma^2)_n\right)$ are not unitary invariant, but the asymptotic freeness of such a Wigner matrices are known by Dykema-Speicher.

Fluctuation of Wigner matrices

On the other hand, since the limit eigenvalue distribution of $\mathcal{E}_2 = \left(\frac{1}{\sqrt{n}}\mathrm{GUE}(\sigma^2)_n\right)$ is, of course, the centered semicircle law of variance σ^2, the first order free cumulants are

$$\kappa_2(\mathcal{E}_2) = \sigma^2, \quad \kappa_n(\mathcal{E}_2) = 0 \ \text{ for } n \neq 2,$$

and all the second order free cumulants will vanish.

Hence, for $\mathcal{E}_1 + \mathcal{E}_2$, we have

$$\begin{cases} \kappa_2(\mathcal{E}_1 + \mathcal{E}_2) = \kappa_2(\mathcal{E}_1) + \kappa_2(\mathcal{E}_2) = (\eta^2 - \sigma^2) + \sigma^2 = \eta^2. \\ \kappa_n(\mathcal{E}_1 + \mathcal{E}_2) = 0 \ \text{ for } n \neq 2, \end{cases}$$

$$\kappa_{1,1}(\mathcal{E}_1 + \mathcal{E}_2) = \kappa_{1,1}(\mathcal{E}_1) + \kappa_{1,1}(\mathcal{E}_2) = -(\eta^2 - \sigma^2) + 0 = \sigma^2 - \eta^2.$$

These are exactly the same as for the case of $\sigma^2 > \eta^2$.

Fluctuation formula of Wigner hermitian matrices

Theorem

The fluctuation moment $\alpha_{p,q}$ of $\left(\frac{1}{\sqrt{n}}\mathrm{Wig}_{\mathbb{C}}(\sigma^2;\eta^2)_n\right)$ is given as

- if p and q are even:

$$\frac{pq}{2(p+q)}\binom{p}{\frac{p}{2}}\binom{q}{\frac{q}{2}}\eta^{p+q},$$

- if p and q are odd:

$$\frac{2pq}{p+q}\binom{p-1}{\frac{p-1}{2}}\binom{q-1}{\frac{q-1}{2}}\eta^{p+q}+\binom{p}{\frac{p-1}{2}}\binom{q}{\frac{q-1}{2}}\left(\frac{\sigma^2}{\eta^2}-1\right)\eta^{p+q}$$

- ortherwise:

$$0.$$

Combinatorial interpletation and real case

(p, q)-annulus and pair

(p, q)-annulus

Let p and q be positive integers. A (p, q)-*annulus* is an annulus in which $1, 2, \ldots, p$ are arranged in clockwise order on the outer circle, and $p+1, p+2, \ldots, p+q$ are arranged in anti-clockwise order on the inner circle.

pair

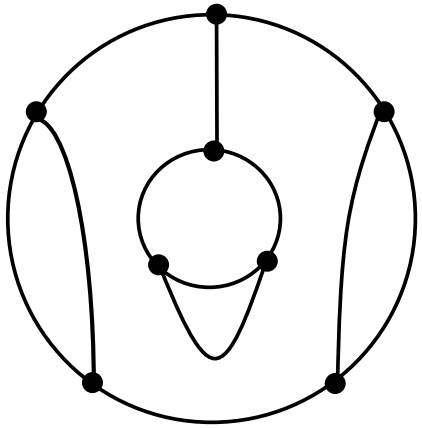
Let (c_i, c_j) with $c_i, c_j \in [p+q]$ be a pair of the points. A pair is called *inner* (resp. *outer*) if both of the points are in the inner (resp. outer) circle.

A pair is called *conected* if it is a pair which connects both points in the inner and the outer circles.

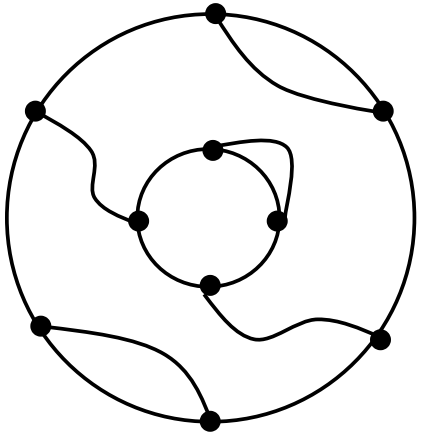
Non-crossing annular parings

Definition

Non-crossing (p, q)*-annular paring* is complete matching by pairs in $[p + q]$ for which we can draw them inside (p, q)-annulus in non-crossing way and it has <u>*at least one*</u> connected pair. We denote by $annNC_2(p, q)$ the set of all non-crossing (p, q)-annular pairings.



$annNC_2(5, 3)$



$annNC_2(6, 4)$

Combinatorial interpretation

The number of $annNC_2(p,q)$ is given by following formula:

$$\left|annNC_2(p,q)\right| = \begin{cases} \dfrac{pq}{2(p+q)}\dbinom{p}{\frac{p}{2}}\dbinom{q}{\frac{q}{2}} & \text{if } p \text{ and } q \text{ are even,} \\ \dfrac{2pq}{p+q}\dbinom{p-1}{\frac{p-1}{2}}\dbinom{q-1}{\frac{q-1}{2}} & \text{if } p \text{ and } q \text{ are odd,} \\ 0 & \text{ortherwise.} \end{cases}$$

Thus, the fluctuation moment $\alpha_{p,q}$ of GUE can be given by

$$\alpha_{p,q} = \left|annNC_2(p,q)\right|.$$

(p, q)-annulus and pair

One of the interesting combinatorial results on non-crossing annular parings is the following lemma:

Lemma

The number of the non-crossing (p, q)-annular parings with exactly c connected pairs is given by

$$c\binom{p}{\frac{p-c}{2}}\binom{q}{\frac{q-c}{2}}.$$

Here the condition $p \equiv q \equiv c \mod 2$ is required, otherwise it is assumed to be 0.

<u>Look the case of p, q being odd in the fluctuation formula!</u>

Combinatorial formula for Wigner complex hermitian

Theorem (Combinatorial form)

The fluctuation moment $\alpha_{p,q}$ of Wigner complex hermitian matrix is given by

$$\alpha_{p,q} = \#\left\{\text{non-crosing } (p,q) \text{ annular pairings}\right\}\eta^{p+q}$$
$$+ \#\left\{\begin{array}{l}\text{non-crosing } (p,q) \text{ annular pairings}\\ \text{with } \textbf{exactly } 1 \textbf{ connected pair}\end{array}\right\}\left(\frac{\sigma^2}{\eta^2}-1\right)\eta^{p+q}$$

Wigner real symmetric matrices

As we have mentioned the first order limit (the moments of the limit distribution) of GOE matrix are unchange and the same as one for GUE matrix, but the fluctuations for GOE matrix are twice their values in GUE case.

This phenomenon was studied for Wishart matrices by Redelmeier via genus expansion. Namely, the fluctuations for a real Wishart matrix are twice their values in the complex case.

Wigner real symmetric matrices

We can find that the same phenomenon will be happened for Wigner real symmetric and complex hermitian matrices.

But we note that since the ratio between the variances at diagonals and at off-diagonals in GOE is $2:1$, the Wigner real symmetric matrix should be decomposed into the sum of GOE and diagonal matrix with taking this ratio into account.

Decomposition of Wigner real symmetric matrix

We denote Winger real symmetric matrix of Gaussian entries as symbolically

$$\begin{pmatrix} \sigma^2 & & \eta^2 \\ & \ddots & \\ \eta^2 & & \sigma^2 \end{pmatrix} \quad \text{or, more simply,} \quad \mathrm{Wig}_{\mathbb{R}}(\sigma^2; \eta^2).$$

The Wigner real symmetric matrices are not orthogonal invariant in general, but only the case of $\mathrm{Wig}_{\mathbb{R}}(2\eta^2; \eta^2)$ is invariant, which corresponds to the Gaussain orthogonal ensembble, and we write $\mathrm{GOE}(\eta^2)$.

Decomposition of Wigner real symmetric matrix

We consider the following independent sum for $\mathrm{Wig}_{\mathbb{R}}(\sigma^2;\eta^2)$:

<u>Case of $\sigma^2 \geq 2\eta^2$</u>

$$\begin{pmatrix} \sigma^2 & & \eta^2 \\ & \ddots & \\ \eta^2 & & \sigma^2 \end{pmatrix} = \begin{pmatrix} 2\eta^2 & & \eta^2 \\ & \ddots & \\ \eta^2 & & 2\eta^2 \end{pmatrix} + \begin{pmatrix} 2(\frac{\sigma^2}{2} - \eta^2) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & 2(\frac{\sigma^2}{2} - \eta^2) \end{pmatrix}$$

$$\mathrm{Wig}_{\mathbb{R}}(\sigma^2;\eta^2) = \mathrm{GOE}(\eta^2) + \mathrm{Dia}(\sigma^2 - 2\eta^2).$$

Case of $\sigma^2 < 2\eta^2$

$$\begin{pmatrix} \sigma^2 & & \eta^2 \\ & \ddots & \\ \eta^2 & & \sigma^2 \end{pmatrix} = \begin{pmatrix} \sigma^2 & & \frac{\sigma^2}{2} \\ & \ddots & \\ \frac{\sigma^2}{2} & & \sigma^2 \end{pmatrix} + \begin{pmatrix} 0 & & \eta^2 - \frac{\sigma^2}{2} \\ & \ddots & \\ \eta^2 - \frac{\sigma^2}{2} & & 0 \end{pmatrix}$$

$$\mathrm{Wig}_{\mathbb{R}}(\sigma^2; \eta^2) = \mathrm{GOE}(\frac{\sigma^2}{2}) + \mathrm{Wig}_{\mathbb{R}}(0; \eta^2 - \frac{\sigma^2}{2})$$

The argument for calculation of the fluctuation moments is parallel to the case of complex.

Fluctuation formula of Wigner real symmetric matrices

Theorem (Fluctuations of Wigner real symmetric)

The fluctuation moment $\alpha_{p,q}$ of $\left(\frac{1}{\sqrt{n}}\mathrm{Wig}_{\mathbb{R}}(\sigma^2;\eta^2)_n\right)$ is given as

- if p and q are even:

$$2\cdot\frac{pq}{2(p+q)}\binom{p}{\frac{p}{2}}\binom{q}{\frac{q}{2}}\eta^{p+q},$$

- if p and q are odd:

$$2\cdot\left\{\frac{2pq}{p+q}\binom{p-1}{\frac{p-1}{2}}\binom{q-1}{\frac{q-1}{2}}+\binom{p}{\frac{p-1}{2}}\binom{q}{\frac{q-1}{2}}\left(\frac{\sigma^2}{2\eta^2}-1\right)\right\}\eta^{p+q}$$

- ortherwise:

$$0.$$

Combinatorial formula for Wigner real symmetric

Theorem (Combinatorial form)

The fluctuation moment $\alpha_{p,q}$ of Wigner real symmetric matrix is given by

$$\begin{aligned}\alpha_{p,q} =& 2^{\#\left\{\text{non-crosing } (p,q) \text{ annular pairings}\right\}} \eta^{p+q} \\ &+ 2^{\#\left\{\begin{array}{l}\text{non-crosing } (p,q) \text{ annular pairings} \\ \text{with } \textbf{exactly } 1 \textbf{ connected pair}\end{array}\right\}} \left(\frac{\sigma^2}{2\eta^2} - 1\right) \eta^{p+q}\end{aligned}$$